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Statistics 100B **Instructor: Nicolas Christou**

Practice 4 - solutions

EXERCISE 1
We know that $X \sim b(n,p)$.

a. The likelihood function is $L(p) = p^x(1-p)^{n-x}$ and the log-likelihood $lnL(p) = xlnp + (n-x)ln(1-p)$. To find the value of p that maximizes the log-likehood we differentiate the previous expression w.r.t. p and set it equal to zero: $\frac{\partial lnL(p)}{\partial p} = \frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow \hat{p} = \frac{x}{n}$.

b. We first find the variance of $\hat{p}$. $Var(\hat{p}) = Var(\frac{x}{n}) = \frac{Var(x)}{n^2} = \frac{np(1-p)}{n^2} \Rightarrow Var(\hat{p}) = \frac{p(1-p)}{n}$. Now we must show that this is equal to the lower bound of the Cramer-Rao inequality. We need the second derivative of the log-pdf function (Bernoulli function $P(Y=y) = p^y(1-p)^{1-y}$: $\frac{\partial^2 lnf(y)}{\partial p^2} = -\frac{y}{p^2} - \frac{1-y}{(1-p)^2}$. Therefore $E(-\frac{\partial^2 lnf(y)}{\partial p^2}) = -\frac{p}{p^2} - \frac{1-p}{(1-p)^2} = -\frac{1}{p(1-p)}$. The Cramer-Rao inequality says that any unbiased estimator must have variance at least: $Var(\hat{\theta}) \geq \frac{1}{nE(-\frac{\partial^2 lnf(y)}{\partial p^2})}$. Therefore $Var(\hat{\theta}) \geq \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n}$, which is the variance of $\hat{p}$. Therefore the mle of p attains the Cramer-Rao inequality.

c. When $n = 10, X = 5$ the log-likelihood function is $lnL(p) = 5lnp + (10-5)ln(1-p)$. Place $lnL(p)$ on the vertical axis and p on the horizontal axis. Then compute $lnL(p)$ for different values of p (try $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$). You will see that the maximum of the log-likelihood function is found when $p = 0.5$ which is the mle $\hat{p} = \frac{x}{n} = \frac{5}{10} = 0.5$.

EXERCISE 2
We need the likelihood fuction which is $L(p) = p^n(1-p)^{\sum_{i=1}^n x_i - n}$. The log-likelihood is $lnL(p) = nlnp + (\sum_{i=1}^n x_i - n)ln(1-p)$ and maximizing it w.r.t. p we get $\frac{\partial lnL(p)}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^n x_i - n}{1-p} = 0 \Rightarrow \frac{n}{p} = \frac{\sum_{i=1}^n x_1 - n}{1-p} \Rightarrow \hat{p} = \frac{1}{\bar{x}}$.

EXERCISE 3
Since μ is known, the maximum likelihood estimator of σ^2 is $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2$. Its expected value is $E(\hat{\sigma^2}) = E(\frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2) = \frac{1}{n}\sum_{i=1}^n E(x_i - \mu)^2) = \frac{n\sigma^2}{n} = \sigma^2$. Therefore it is unbiased.

EXERCISE 4
We must first compute the variance of $\hat{\mu}$ and $\bar{x}$. They are: $Var(\hat{\mu}) = Var(\frac{x_1 + 2x_2 + x_3}{4}) = \frac{6\sigma^2}{16}$ and $Var(\bar{x}) = \frac{\sigma^2}{3}$. The relative efficieny of $\hat{\mu}$ with respect to $\bar{x}$ is $\frac{\bar{x}}{\hat{\mu}} = \frac{\frac{\sigma^2}{3}}{\frac{6\sigma^2}{16}} = \frac{8}{9}$.

EXERCISE 5
We are given $X_1 \sim N(\mu, \sigma_1)$, and$X_2 \sim N(\mu, \sigma_2)$. Also $0 \leq w \leq 1$.

a. $E(w\bar{x}_1 + (1-w)\bar{x}_2) = wE(\bar{x}_1) + (1-w)E(\bar{x}_2) = w\mu + (1-w)\mu = \mu$.

b. We want to minimize the variance of $w\bar{x}_1 + (1-w)\bar{x}_2$.
minimize $Var(w\bar{x}_1 + (1-w)\bar{x}_2)$ w.r.t w
minimize $w^2\frac{\sigma_1^2}{n} + (1-w)^2\frac{\sigma_2^2}{n}$ w.r.t. w
Or $2w\frac{\sigma_1^2}{n} - 2(1-w)\frac{\sigma_2^2}{n} = 0 \Rightarrow w\sigma_1^2 - (1-w)\sigma_2^2 = 0 \Rightarrow w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$.

EXERCISE 6
Find the mle of the parameter λ of the Poisson distribution:
The likelihood function is $L(x_1, \cdots, x_n; \lambda) = \frac{e^{-\lambda n}\lambda^{\sum_{i=1}^n x_i}}{x_1! \cdots x_n!}$, and the log-likelihood function is $lnL(x_1, \cdots, x_n; \lambda) = -\lambda n + \sum_{i=1}^n x_1 ln\lambda - ln(x_1! \cdots x_n)$. We want to find the value of λ that maximizes the previous expression:
$\frac{\partial lnL}{\partial \lambda} = -n + \frac{\sum_{i=1}^n x_i}{\lambda} = 0 \Rightarrow \hat{\lambda} = \bar{x}$.

EXERCISE 7
We need to show that $\frac{x}{n}$ is unbiased estimator of p and that $Var(\frac{x}{n}) = 0$ as $n \to \infty$.
$E(\frac{x}{n}) = \frac{np}{n} = p$, so it is unbiased.
$Var(\frac{x}{n}) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$ which is equal to zero as $n \to \infty$.
Therefore $\frac{x}{n}$ is a consistent estimator of p.

EXERCISE 8
We know that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$. We need to show that s^2 is unbiased estimator of σ^2 and that $vars^2 = 0$ as $n \to \infty$.
$E(\frac{(n-1)s^2}{\sigma^2}) = n-1 \Rightarrow Es^2 = \sigma^2 \frac{n-1}{n-1} = \sigma^2$, so it is unbiased.
$Var(\frac{(n-1)s^2}{\sigma^2}) = 2(n-1) \Rightarrow Vars^2 = \frac{2(n-1)}{(n-1)^2}\sigma^4 \Rightarrow Vars^2 = \frac{2\sigma^4}{n-1}$ which is equal to zero as $n \to \infty$.
Therefore s^2 is a consistent estimator of σ^2.

EXERCISE 9
The esimate of p is $\hat{p} = \frac{X}{n}$, and the estimate of σ^2 is $\hat{\sigma}^2 = \frac{\hat{p}(1-\hat{p})}{n} = \frac{\frac{X}{n}(1-\frac{X}{n})}{n}$. To see whether $\hat{\sigma}^2$ is an unbiased estimator of σ^2 we need to find its expected value. We will need $E(X) = np$, and $E(X^2) = \sigma^2 + \mu^2 = np(1-p) + (np)^2 = np - np^2 + n^2p^2$.

$$E(\hat{\sigma}^2) = E\left(\frac{\frac{X}{n}(1-\frac{X}{n})}{n}\right) =$$

$$E\left(\frac{X}{n^2} - \frac{X^2}{n^3}\right) = \frac{1}{n^2}E(X) - \frac{1}{n^3}E(X^2) =$$

$$\frac{1}{n^2}np - \frac{1}{n^3}(np - np^2 + n^2p^2) = \frac{p}{n} - \frac{p}{n^2} + \frac{p^2}{n^2} - \frac{p^2}{n} =$$

$$(\frac{1}{n} - \frac{1}{n^2})(p-p^2) = (\frac{1}{n} - \frac{1}{n^2})p(1-p) \Rightarrow$$

$$E(\hat{\sigma}^2) = (1-\frac{1}{n})\frac{p(1-p)}{n}.$$

It is not unbiased but we can multiply it by the reciprocal of $1-\frac{1}{n}$ which is $\frac{n}{n-1}$ to make it unbiased. Therefore, $\frac{n}{n-1}\hat{\sigma}^2$ is unbiased estimator of $\frac{p(1-p)}{n}$.

EXERCISE 10
We are given that $X_i \sim N(\mu, \frac{\sigma}{\sqrt{w_i}})$. The likelihood function of $X_1, X_2, \cdots, X_n$ is

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} \Pi_{i=1}^n \sqrt{w_i}\ e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n w_i(x_i-\mu)^2}.$$

And the log-likelihood is

$$lnL = -\frac{n}{2}ln(2\pi\sigma^2) + ln\Pi_{i=1}^n\sqrt{w_i} - \frac{1}{2\sigma^2}\sum_{i=1}^n w_i(x_i-\mu)^2.$$

First we find the mle of μ:

$$\frac{\partial lnL}{\partial \mu} = \sum_{i=1}^n w_i(x_i-\mu) = 0 \Rightarrow \sum_{i=1}^n w_i x_i = \sum_{i=1}^n w_i\mu \Rightarrow \hat{\mu} = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}.$$

Now the mle of σ^2:

$$\frac{\partial lnL}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n w_i(x_i-\mu)^2 = 0 \Rightarrow n = \frac{1}{\sigma^2}\sum_{i=1}^n w_i(x_i-\hat{\mu})^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n w_i(x_i-\hat{\mu})^2.$$

where $\hat{\mu} = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$.

EXERCISE 11
Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from $N(\mu, \sigma)$.

a. Which of the following estimates is unbiased? Show all your work.

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}, \quad S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

Answer:
The sample variance (S^2) is unbiased (please see class notes). The mle of σ^2 is biased. We can write it as follows:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n} = \frac{\sigma^2}{n}\frac{(n-1)S^2}{\sigma^2}.$$

Therefore,

$$E(\hat{\sigma}^2) = E\left(\frac{\sigma^2}{n}\frac{(n-1)S^2}{\sigma^2}\right) = \frac{n-1}{n}\sigma^2$$

b. Which of the estimates of part (a) has the smaller MSE? The MSE is equal to: $MSE = Var(\hat{\theta}) + B^2$. We need to find the variance of each of the estimators. We have shown in class that $Var(S^2) = \frac{2\sigma^4}{n-1}$. To find the variance of $\hat{\sigma}^2$:

$$Var(\hat{\sigma}^2) = Var\left(\frac{\sigma^2}{n}\frac{(n-1)S^2}{\sigma^2}\right) = \frac{2(n-1)}{n^2}\sigma^4.$$

The bias of S^2 is zero (it s unbiased). Therefore,

$$MSE(S^2) = Var(S^2) = \frac{2\sigma^4}{n-1}.$$

The bias of $\hat{\sigma}^2$ is equal to:

$$B = E(\hat{\sigma}^2) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{n}.$$

And its MSE s equal to:

$$MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) + B^2 = \frac{2(n-1)}{n^2}\sigma^4 + \frac{\sigma^4}{n^2} = \frac{2n-1}{n^2}\sigma^4.$$

We can easily see that

$$\frac{MSE(\hat{\sigma}^2)}{MSE(S^2)} < 1$$

EXERCISE 12

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from a normal population with mean zero and unknown variance σ^2.

a. Find the maximum likelihood estimate of σ^2.
Answer:

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\sigma^2}}$$

$$lnL = -\frac{n}{2}ln(2\pi\sigma^2) - \frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\sigma^2}$$

$$\frac{\partial lnL}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n} x_i^2 = 0$$

Solve for σ^2 to get: $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n}$.

b. Show that the estimate of part (a) is unbiased estimator of σ^2.
Answer:

$$E(\hat{\sigma}^2) = E\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = \frac{\sigma^2}{n}E\left(\sum_{i=1}^{n}(\frac{x_i - 0}{\sigma})^2\right) = \frac{\sigma^2}{n}E(\chi_n^2) = \frac{\sigma^2}{n}n = \sigma^2.$$

c. Find the variance of the estimate of part (a). Is it consistent?
Answer:

$$Var(\hat{\sigma}^2) = Var\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = \frac{\sigma^4}{n^2}Var\left(\sum_{i=1}^{n}(\frac{x_i - 0}{\sigma})^2\right) = \frac{\sigma^4}{n^2}Var(\chi_n^2) = \frac{\sigma^4}{n^2}2n = \frac{2\sigma^4}{n}.$$

It is consistent because it is unbiased and its variance is equal to zero as $n \to \infty$.
$E(\hat{\sigma}^2) = \sigma^2$ and

$$\lim_{n\to\infty}\frac{2\sigma^4}{n} = 0$$

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d. Show that the variance of the estimate of part (a) is equal to the Cramer-Rao lower bound.
Answer:

$$lnf(x) = ln(2\pi\sigma^2)^{-\frac{1}{2}} - \frac{1}{2}(\frac{x-0}{\sigma})^2 = -\frac{1}{2}ln(2\pi) - \frac{1}{2}ln(\sigma^2) - \frac{1}{2}\frac{x^2}{\sigma^2}.$$

We find now the first and second derivatives w.r.t. σ^2.

$$\frac{\partial lnf(x)}{\partial\sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}x^2.$$

$$\frac{\partial^2 lnf(x)}{\partial(\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{x^2}{\sigma^6}.$$

$$E\left(\frac{\partial^2 lnf(x)}{\partial(\sigma^2)^2}\right) = E\left(\frac{1}{2\sigma^4} - \frac{x^2}{\sigma^6}\right) = \frac{1}{2\sigma^4} - \frac{EX^2}{\sigma^6} = -\frac{1}{2\sigma^4}.$$

Therefore,

$$\frac{1}{-nE\left(\frac{\partial^2 lnf(x)}{\partial(\sigma^2)^2}\right)} = \frac{1}{-n(-\frac{1}{2\sigma^4})} = \frac{2\sigma^4}{n}.$$

Therefore, $\hat{\sigma}^2$ is MVUE.