University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Practice 4

EXERCISE 1
Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables from a Poisson distribution with parameter λ. We know that the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x}$.

a. Find the variance of $\hat{\lambda}$.

b. Is $\hat{\lambda}$ an MVUE?

c. Is $\hat{\lambda}$ a consistent estimator of λ?

EXERCISE 2
Suppose that two independent random samples of n_1 and n_2 observations are selected from two normal populations. Further, assume that the populations possess a common variance σ^2 which is unknown. Let the sample variances be S_1^2 and S_2^2 for which $E(S_1^2) = \sigma^2$ and $E(S_2^2) = \sigma^2$.

a. Show that the pooled estimator of σ^2 that we derived in class below is unbiased.

$$S^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

b. Find the variance of S^2.

EXERCISE 3
Suppose $Y_i = \beta_1 x_i + \epsilon_i$. This is called regression through the origin (no intercept). In this simple regression model equation x_i is non-random, β_1 is a parameter (unknown), and $\epsilon_i \sim N(0, \sigma)$.

a. Find the mean of Y_i.

b. Find the variance of Y_i.

c. What distribution does Y_i follow? Write the pdf of Y_i.

d. Write the likelihood function based on n observations of Y and x.

e. Find the maximum likelihood estimate of β_1. Denote them with $\hat{\beta}_1$.

f. Show that $\hat{\beta}_1$ is unbiased estimator and β_1.

g. Find the variance of $\hat{\beta}_1$.

h. What is the distribution of $\hat{\beta}_1$?

i. Find the maximum likelihood estimate of σ^2.

EXERCISE 4
Consider a random sample $X_1, X_2, \cdots, X_n$ from the probability density function

$$f(x; \theta) = \frac{1 + \theta x}{2}, \quad -1 \leq x \leq 1$$

a. Find $\hat{\theta}$, the method of moments estimator of θ.

b. Is $\hat{\theta}$ unbiased?

c. Show that $Var(\hat{\theta}) = \frac{3-\theta^2}{n}$.

d. Is $\hat{\theta}$ a consistent estimator of θ?

EXERCISE 5
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ be random variables each one having gamma distribution with parameters α, β. The probability density function, mean, and variance of the gamma distribution are given below:

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}, \quad \alpha, \beta > 0, x \geq 0.$$

$E(X) = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$. Use the method of moments to estimate α and β.

b. Let $X_1, X_2, \ldots, X_n$ be random variables each one having uniform distribution on the interval $(0, 3\theta)$. Derive the method of moments estimator of θ.

c. Is the estimator from part (b) consistent? Explain.

EXERCISE 6
Let $X_1, X_2, \ldots, X_n$ be random variables each one having the following distribution (this is one of the Pareto family of distributions). Suppose the parameter $\theta > 0$ is unknown.

$$f(x) = \begin{cases} 3\theta^3 x^{-4}, & x \geq \theta \\ 0, & \text{elsewhere} \end{cases}$$

Consider the estimator $\hat{\theta} = X_{(1)} = \min(X_1, X_2, \ldots, X_n)$.

a. Show that the probability density function of $X_{(1)}$ is $g_1(x) = 3n\theta^{3n}x^{-3n-1}$.

b. Show that the expected value of the estimator above is $E(\hat{\theta}) = \frac{3n\theta}{3n-1}$.

c. Show that the bias of $\hat{\theta}$ is $B = \frac{\theta}{3n-1}$.

EXERCISE 7
For the regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ show that

a. $\sum_{i=1}^{n} e_i = 0$.

b. $Cov(\bar{Y}, \hat{\beta}_1) = 0$ where $\bar{Y}$ is the sample mean of the y values, and $\hat{\beta}_1$ is the estimate of β_1.

EXERCISE 8
Consider the regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$. Find $cov(e_i, e_j)$.