EXERCISE 1
Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a Poisson distribution with parameter $\lambda$. We know that the maximum likelihood estimate of $\lambda$ is $\hat{\lambda} = \bar{x}$.

a. Find the variance of $\hat{\lambda}$.
Answer:
$$Var(\bar{X}) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{n\lambda}{n^2} = \frac{\lambda}{n}.$$

b. Is $\hat{\lambda}$ an MVUE?
Answer: We need to find the lower bound of the Cramer-Rao inequality:
$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \Rightarrow \ln f(x) = x\ln\lambda - \lambda - \ln x!$$

Let’s find the first and second derivatives w.r.t. $\lambda$.
$$\frac{\partial \ln f(x)}{\partial \lambda} = \frac{x}{\lambda} - 1 \quad \text{and} \quad \frac{\partial^2 \ln f(x)}{\partial \lambda^2} = -\frac{x}{\lambda^2}.$$

Therefore,
$$\frac{1}{-nE(\frac{\partial^2 \ln f(x)}{\partial \lambda^2})} = \frac{1}{-nE(-\frac{x}{\lambda^2})} = \frac{\lambda^2}{\lambda n} = \frac{\lambda}{n}.$$

Therefore, $\hat{\lambda}$ is MVUE.

c. Is $\hat{\lambda}$ a consistent estimator of $\lambda$?
Answer:
It is consistent because it is unbiased and its variance is equal to zero as $n \to \infty$.
$$E(\hat{\lambda}) = \lambda$$ and
$$\lim_{n \to \infty} \frac{\lambda}{n} = 0.$$

EXERCISE 2
Suppose that two independent random samples of $n_1$ and $n_2$ observations are selected from two normal populations. Further, assume that the populations possess a common variance $\sigma^2$ which is unknown. Let the sample variances be $S_1^2$ and $S_2^2$ for which $E(S_1^2) = \sigma^2$ and $E(S_2^2) = \sigma^2$.

a. Show that the pooled estimator of $\sigma^2$ that we derived in class below is unbiased.
$$S^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Answer:
Both $S_1^2$ and $S_2^2$ are unbiased estimators of $\sigma^2$. Therefore:
$$E(S^2) = E\left(\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}\right) = \frac{(n_1 - 1)E(S_1^2) + (n_2 - 1)E(S_2^2)}{n_1 + n_2 - 2} = \frac{(n_1 - 1)\sigma^2 + (n_2 - 1)\sigma^2}{n_1 + n_2 - 2} = \sigma^2.$$
b. Find the variance of $S^2$.
Answer:
We use the result that \( \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \) as follows:
\[
(n_1 + n_2 - 2)S^2 = (n_1 - 1)S^2_1 + (n_2 - 1)S^2_2
\]
Or
\[
\frac{(n_1 + n_2 - 2)S^2}{\sigma^2} = \frac{(n_1 - 1)S^2_1}{\sigma^2} + \frac{(n_2 - 1)S^2_2}{\sigma^2}
\]
Therefore,
\[
\frac{(n_1 + n_2 - 2)S^2}{\sigma^2} \sim \chi^2_{n_1 + n_2 - 2}
\]
Finally,
\[
Var \left( \frac{(n_1 + n_2 - 2)S^2}{\sigma^2} \right) = 2(n_1 + n_2 - 2) \Rightarrow Var(S^2) = \frac{2(n_1 + n_2 - 2)\sigma^4}{(n_1 + n_2 - 2)^2} = \frac{2\sigma^4}{n_1 + n_2 - 2}.
\]
EXERCISE 3

Suppose $Y_i = \beta_1 x_i + \epsilon_i$. In this simple regression model through the origin $x_i$ is non-random, $\beta_1$ is unknown parameter, and $\epsilon_i \sim N(0, \sigma)$.

a. Find the mean of $Y_i$.

b. Find the variance of $Y_i$.

c. What distribution does $Y_i$ follow? Write the pdf of $Y_i$.

d. Write down the likelihood function based on $n$ observations of $Y$ and $x$.

e. Find the maximum likelihood estimate of $\beta_1$. Denote it with $\hat{\beta}_1$.

f. Show that $\hat{\beta}_1$ is unbiased estimator of $\beta_1$.

g. Find the variance of $\hat{\beta}_1$.

h. What is the distribution of $\hat{\beta}_1$?

i. Find the maximum likelihood estimate of $\sigma^2$.

Solution:

Let $Y_i = \beta_1 x_i + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma)$. The $x_i$’s are non-random.

a. $E(Y_i) = \beta_1 x_i$.

b. $\text{var}(Y_i) = \sigma^2$

c. $Y_i \sim N(\beta_1 x_i, \sigma)$. And its pdf function is:

$$f(y_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y_i - \beta_1 x_i}{\sigma} \right)^2}$$

d. $L = (\sigma^2 2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} \sum_{i=1}^{n} \left( \frac{y_i - \beta_1 x_i}{\sigma} \right)^2}$

e. From part (d) we can write the log-likelihood function:

$$\ln(L) = -\frac{n}{2} \ln(\sigma^2 2\pi) - \frac{1}{2} \sum_{i=1}^{n} \left( \frac{y_i - \beta_1 x_i}{\sigma} \right)^2.$$  

To find the estimate of $\beta_1$ we take the partial derivative of the log-likelihood w.r.t. to $\beta_1$, set it equal to zero and solve:

$$\frac{\partial \ln(L)}{\partial \beta_1} = 2 \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_1 x_i)x_i = 0$$

Solving for $\beta_1$ we get:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}.$$  

f. The previous estimate is unbiased because:

$$E(\hat{\beta}_1) = \frac{\sum_{i=1}^{n} x_i E(y_i)}{\sum_{i=1}^{n} x_i^2} = \beta_1.$$  

g. The variance of this estimate is:

$$\text{var}(\hat{\beta}_1) = \frac{\text{var} \left( \sum_{i=1}^{n} \frac{x_i y_i}{x_i^2} \right)}{\sum_{i=1}^{n} x_i^2} = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}.$$  

h. Distribution of $\hat{\beta}_1$:

$$\hat{\beta}_1 \sim N(\beta_1, \frac{\sigma}{\sqrt{\sum_{i=1}^{n} x_i^2}}).$$  

i. Maximum likelihood of $\sigma^2$:

$$\frac{\partial \ln(L)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \beta_1 x_i)^2 = 0 \Rightarrow \sigma^2 = \frac{\sum_{i=1}^{n} (y_i - \hat{\beta}_1 x_i)^2}{n}.$$
EXERCISE 4

a. We need to find first the mean of $X$:

$$E(X) = \int_{-1}^{1} x \frac{1 + \theta x}{2} dx = \left[ \frac{x^2}{4} + \frac{\theta x^3}{6} \right]_{-1}^{1} = \frac{\theta}{3}.$$

Therefore the method of moments estimator of $\theta$ is $\hat{\theta} = \bar{x} \Rightarrow \hat{\theta} = 3\bar{x}$.

b. Yes, $\hat{\theta}$ is unbiased because $E(\hat{\theta}) = 3E(\bar{x}) = 3\mu = 3\frac{\theta}{3} = \theta$.

c. We need to find first the variance of $X$:

$$\text{var}(X) = EX^2 - (E(X))^2 = \int_{-1}^{1} x^2 \frac{1 + \theta x}{2} dx - \left( \frac{\theta}{3} \right)^2 = \left[ \frac{x^3}{6} + \frac{\theta x^4}{8} \right]_{-1}^{1} - \left( \frac{\theta}{3} \right)^2 = 1 - \left( \frac{\theta}{3} \right)^2 = \frac{3 - \theta^2}{9}.$$

Therefore, $\text{var}(\hat{\theta}) = \text{var}(3\bar{x}) = 9 \frac{\text{var}(\bar{x})}{n} = 9 \frac{3\theta^2}{9n} = \frac{3\theta^2}{n}$.

d. Yes, $\hat{\theta}$ is consistent because it is unbiased, and its variances goes to zero as $n$ goes to infinity.

EXERCISE 5

a. Using the method of moments we get:

$$E(X) = \alpha \beta \Rightarrow \alpha \bar{x} = \beta \Rightarrow \beta = \frac{\bar{x}}{\alpha},$$

and

$$\sigma^2 = EX^2 - (E(X))^2 = \alpha \beta^2 \Rightarrow \frac{\sum_{i=1}^{n} x_i^2}{n} - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2 = \alpha \beta^2.$$

From the two expressions above we get:

$$\alpha \beta^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2,$$

$$\frac{\bar{x}^2}{\alpha^2} = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2,$$

$$\hat{\alpha} = \frac{\bar{x}^2}{\sum_{i=1}^{n} x_i^2 - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2} = \frac{\bar{x}^2}{\sum_{i=1}^{n} x_i^2 - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2} \Rightarrow \hat{\alpha} = \frac{n \bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

and $\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n \bar{x}}$.

b. $E(X) = \frac{3\theta}{2} \Rightarrow \hat{\theta} = \frac{2}{3} \bar{x}$.

c. The estimator of part (b) is consistent because:

1. It is unbiased: $E(\hat{\theta}) = \frac{2}{3} \mu = \frac{2}{3} \frac{\theta}{3} = \theta$.
2. $\text{var}(\hat{\theta}) = \frac{4}{9} \sigma^2 = \frac{4 \theta^2}{9} \frac{1}{2n} \Rightarrow \theta^2 \frac{4}{9n} \rightarrow 0$ as $n \rightarrow \infty$.

EXERCISE 6

We need to find first the cdf of $X$:

$$F(X) = \int_{\theta}^{x} 3\theta^3 u^{-4} du = \frac{[3\theta^4 u^{-3}]_{\theta}^{x}}{3} = 1 - \theta^3 x^{-3}.$$

a. Therefore the pdf of $X(1)$ is:

$$f_1(x) = n (1 - F(x))^{n-1} f(x) = n \left( \theta^3 x^{-3} \right)^{n-1} 3\theta^3 x^{-4} = 3n \theta^3 x^{-3} n^{-1}.$$
b. The mean of $X(1)$:

$$EX(1) = \int_0^\infty x^{3n-3}x^{-3n-1}dx = 3n\theta^{3n}x^{-3n+1}^{3n} = \frac{3n\theta}{3n-1}.$$ 

c. $B = E\theta - \theta = \frac{3n\theta}{3n-1} - \theta = \frac{\theta}{3n-1}.$

**EXERCISE 7**

Solution:

a. We must show that the sum of the residuals is equal to zero.

$$\sum_{i=1}^n e_i = \sum_{i=1}^n (y_i - \hat{y}_i) = \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) =$$

$$\sum_{i=1}^n (y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 x_i) = \sum_{i=1}^n (y_i - \bar{y}) - \hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x}) = 0.$$ 

b. The $\text{cov}(\hat{Y}, \hat{\beta}_1) = 0$ because:

$$\text{cov}(\hat{Y}, \hat{\beta}_1) = Cov\left(\frac{1}{n} \sum_{i=1}^n Y_i \sum_{i=1}^n (x_i - \bar{x})Y_i \sum_{i=1}^n (x_i - \bar{x})^2\right) =$$

$$= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \left[\text{Var}(Y_1) + (x_2 - \bar{x})\text{Var}(Y_2) + \cdots + (x_n - \bar{x})\text{Var}(Y_n)\right] =$$

$$= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \frac{\sigma^2}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = 0.$$ 

Note: $\sum_{i=1}^n (x_i - \bar{x}) = 0$ and $\text{cov}(Y_i, Y_j) = 0$ because the $Y$’s are independent.

**EXERCISE 8**

Solution:

$$\text{cov}(e_i, e_j) = \text{cov}(Y_i - \hat{Y}_i, Y_j - \hat{Y}_j) = \text{cov}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i, Y_j - \hat{\beta}_0 - \hat{\beta}_1 x_j) =$$

$$\text{cov}(Y_i - \bar{Y} - \hat{\beta}_1 \bar{x}, Y_j - \bar{Y} - \hat{\beta}_1 \bar{x}) =$$

$$= \text{cov}(Y_i, Y_j) - \text{cov}(Y_i, \bar{Y}) - (x_j - \bar{x})\text{cov}(\bar{Y}, \hat{\beta}_1)$$

$$-\text{cov}(\bar{Y}, Y_j) - \text{var}(\bar{Y}) + (x_j - \bar{x})\text{cov}(\bar{Y}, \hat{\beta}_1)$$

$$= 0 - \frac{\sigma^2}{n} \frac{(x_j - \bar{x})(x_j - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot$$

$$\frac{\sigma^2}{n} + \frac{\sigma^2}{n} + 0$$

$$= \sigma^2 \left[ - \frac{1}{n} \frac{(x_j - \bar{x})(x_j - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \right].$$