Answer the following questions:

a. We discussed in class today the multinomial probability distribution and its joint moment generating function. Here is a note on the multinomial distribution: A sequence of \( n \) independent experiments is performed and each experiment can result in one of \( r \) possible outcomes with probabilities \( p_1, p_2, \ldots, p_r \) with \( \sum_{i=1}^{r} p_i = 1 \). Let \( X_i \) be the number of the \( n \) experiments that result in outcome \( i, i = 1, 2, \ldots, r \). Then, \( P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{n_1!n_2! \cdots n_r!} p_1^{x_1}p_2^{x_2} \cdots p_r^{x_r} \). The joint moment generating function of the multinomial distribution is given by \( M_X(t) = (p_1e^{t_1} + p_2e^{t_2} + \ldots + p_re^{t_r})^n \). Use properties of joint moment generating functions to find the probability distribution of \( X_1 \).

b. Refer to question (a). Use the joint moment generating function of the multinomial distribution and the theorem and corollary on handout #10, page 1 to find the mean and variance of \( X_1 \).

c. Refer to question (a). Show that \( \text{cov}(X_i, X_j) = -np_ip_j \). Give an intuitive explanation of the negative sign.

d. Let \( X \) and \( Y \) be independent normal random variables, each with mean \( \mu \) and standard deviation \( \sigma \).
   1. Consider the random quantities \( X + Y \) and \( X - Y \). Find the moment generating function of \( X + Y \) and the moment generating function of \( X - Y \).
   2. Find the joint moment generating function of \( (X + Y, X - Y) \).
   3. Are \( X + Y \) and \( X - Y \) independent? Explain your answer using moment generating functions.