University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Quiz 2

EXERCISE 1
Let $U \sim N(0,1)$, $V \sim \chi^2_n$, and U and V are independent. Let $t = \frac{U}{\sqrt{\frac{V}{n}}}$ and $W = V$. Find the joint pdf of t and W and then integrate the joint w.r.t. to W to show that the probability density function of the t distribution with $df = n$ degrees of freedom is

$$f(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < t < \infty.$$

EXERCISE 2
Suppose $Q_1, \ldots, Q_k$ are independent. Let $Q_1 \sim \chi^2_{p_1}(\theta_1), \ldots, Q_k \sim \chi^2_{p_k}(\theta_k)$, where $p_1, \ldots, p_k$ are the degrees of freedom and $\theta_1, \ldots, \theta_k$ are the non-centrality parameters. Find the mean and variance of $Y = Q_1 + \ldots + Q_k$.

EXERCISE 3
Answer the following questions:

a. We discussed in class today the multinomial probability distribution and its joint moment generating function. Here is a note on the multinomial distribution: A sequence of n independent experiments is performed and each experiment can result in one of r possible outcomes with probabilities $p_1, p_2, \ldots, p_r$ with $\sum_{i=1}^r p_i = 1$. Let X_i be the number of the n experiments that result in outcome i, $i = 1, 2, \ldots, r$. Then, $P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{n_1! n_2! \cdots n_r!} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$. The joint moment generating function of the multinomial distribution is given by $M_{\mathbf{X}}(\mathbf{t}) = (p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_r e^{t_r})^n$. Use properties of joint moment generating functions to find the probability distribution of X_1.

b. Refer to question (a). Find the mean of $\mathbf{X}$ and the variance covariance matrix of $\mathbf{X}$, where $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$.

EXERCISE 4
Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with mean $i\theta$. For example, $E(X_1) = \theta, E(X_2) = 2\theta$, etc. Suppose an estimate of of θ is $\hat{\theta} = \sum_{i=1}^n (\frac{X_i}{ni})$.

a. Find the distribution of $\hat{\theta}$.

b. Find $E[\hat{\theta}^{-1}]$.

c. Find the MSE of $c\hat{\theta}^{-1}$ as an estimator of θ^{-1}, and find c that minimizes that MSE.

EXERCISE 5
Let $X_1, \ldots, X_n$ be i.i.d. random variables with $f(x) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha}, \alpha > 0, \theta > 0, 0 \leq x \leq \theta$. Assume that θ is known. Use the factorization theorem to show that $\Pi_{i=1}^n X_i$ is a sufficient statistics for α.