Answer the following questions:

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with $\alpha$ known. Let $\hat{\beta}$ be the MLE of $\beta$. Is $\frac{1}{\hat{\beta}}$ an unbiased estimator of $\frac{1}{\beta}$? If not, adjust it to be unbiased.

2. Let $X_1, X_2, \ldots, X_m$ i.i.d. random variables with $X_i \sim N(0, \sigma)$.
   Let $Y_1, Y_2, \ldots, Y_m$ i.i.d. random variables with $Y_i \sim N(0, \sigma)$.
   Let $Z_1, Z_2, \ldots, Z_l$ i.i.d. random variables with $Z_i \sim N(0, \sigma)$.
   These three random samples are independent. Find the maximum likelihood estimate of $\sigma^2$. Is it unbiased?

3. Refer to question (2). Is the MLE of $\sigma^2$ an efficient estimator?

4. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables with $Y_i \sim \exp(\lambda)$. Find the mean and variance of $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$.

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a uniform distribution on $(0, \theta)$. We discussed in class that the MLE of $\theta$ is $\hat{\theta} = \max(X_1, X_2, \ldots, X_n)$ (also denoted as $X_{(n)}$). We also found that $\frac{n+1}{n} \hat{\theta}$ is an unbiased estimator of $\theta$. But is the fraction $\frac{n+1}{n}$ the best multiple of $\hat{\theta}$? To answer this question consider the estimator $c\hat{\theta}$. Find the value of $c$ that minimizes the mean squared error.