

Quiz 8

Answer the following questions:

- a. Suppose  $X_1, \dots, X_n$  are i.i.d.  $\text{Poisson}(\lambda)$ . Let  $S = \sum_{i=1}^n X_i$ . Show that  $(X_1 = x_1, \dots, X_n = x_n)$  conditioned on  $S = N$  follows the multinomial distribution with parameters  $S$  and  $(\frac{1}{n}, \dots, \frac{1}{n})$ .  
Hint 1: Find the joint pmf of  $X_i$ 's. Given that  $S = N$ ,  $\sum_i X_i = N$ . Use this result in the joint pmf of the  $X_i$ 's.  
Hint 2: Continue by expressing the conditional pmf as the ratio of the joint and the marginal pmf's.  
Note on the multinomial probability distribution:  
A sequence of  $n$  independent experiments is performed and each experiment can result in one of  $r$  possible outcomes with probabilities  $p_1, p_2, \dots, p_r$  with  $\sum_{i=1}^r p_i = 1$ . Let  $X_i$  be the number of the  $n$  experiments that result in outcome  $i$ ,  $i = 1, 2, \dots, r$ . Then,  $P(X_1 = x_1, X_2 = x_2, \dots, X_r = x_r) = \frac{n!}{n_1! n_2! \dots n_r!} p_1^{x_1} p_2^{x_2} \dots p_r^{x_r}$ .
- b. Refer to question (a). Suppose we know that  $E[X(X-1)] = \lambda^2$  and  $\text{var}[X(X-1)] = 2\lambda^2 + 4\lambda^3$ . Please explain how you would verify these two results. Let  $T_1 = \frac{1}{n} \sum_{i=1}^n X_i(X_i - 1)$ . Show that  $T_1$  is unbiased estimator for  $\lambda^2$  and compute its variance.
- c. Refer to question (a). Let  $T_2 = E[T_1|S]$ . Show that  $T_2 = \frac{S(S-1)}{n^2}$ . Note that  $E[X_i|S] = S \frac{1}{n}$  and  $\text{var}(X_i|S) = S \frac{1}{n} (1 - \frac{1}{n})$ . Is  $T_2$  unbiased estimator of  $\lambda^2$ ? Find  $\text{var}(T_2)$ .
- d. Let  $X_1, \dots, X_n$  be i.i.d. random variables with  $f(x) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha}$ ,  $\alpha > 0, \theta > 0, 0 \leq x \leq \theta$ . Assume that  $\theta$  is known. Use the factorization theorem to show that  $\prod_{i=1}^n X_i$  is a sufficient statistics for  $\alpha$ .