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Statistics 100B **Instructor: Nicolas Christou**

Properties of estimators

Unbiased estimators:
Let $\hat{\theta}$ be an estimator of a parameter θ. We say that $\hat{\theta}$ is an unbiased estimator of θ if

$$E(\hat{\theta}) = \theta$$

Examples:
Let $X_1, X_2, \cdots, X_n$ be an i.i.d.sample from a population with mean μ and standard deviation σ. Show that $\bar{X}$ and S^2 are unbiased estimators of μ and σ^2 respectively.

Efficient estimators:

Cramér-Rao inequality:

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample from a distribution that has pdf $f(x)$ and let $\hat{\theta}$ be an unbiased estimator of a parameter θ of this distribution. We will show that the variance of $\hat{\theta}$ is at least:

$$var(\hat{\theta}) \geq \frac{1}{nE(\frac{\partial lnf(x)}{\partial \theta})^2} = \frac{1}{nI(\theta)} \qquad \text{or} \qquad var(\hat{\theta}) \geq \frac{1}{-nE(\frac{\partial^2 lnf(x)}{\partial \theta^2})} = \frac{1}{nI(\theta)},$$

where $I(\theta)$ is the information on one observation.

Theorem:

We say that $\hat{\theta}$ is an efficient estimator of θ if $\hat{\theta}$ is an unbiased estimator of θ and if

$$var(\hat{\theta}) = \frac{1}{nE(\frac{\partial lnf(x)}{\partial \theta})^2} = \frac{1}{nI(\theta)}$$

In other words, if the variance of $\hat{\theta}$ attains the minimum variance of the Cramér-Rao inequality we say that $\hat{\theta}$ is an efficient estimator of $\hat{\theta}$.

Example:

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample from a normal population with mean μ and standard deviation σ. Show that $\bar{X}$ is a minimum variance unbiased estimator of μ. Verify that the result can be obtained using $I(\theta) = E(\frac{\partial lnf(x)}{\partial \theta})^2 = -E(\frac{\partial^2 lnf(x)}{\partial \theta^2}) = \text{var}(S)$, where S is the score function (see page 6).

Relative efficiency:
If $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators of a parameter θ we say that $\hat{\theta}_1$ is relatively more efficient if $var(\hat{\theta}_1) < var(\hat{\theta}_2)$. We use the ratio

$$\frac{var(\hat{\theta}_1)}{var(\hat{\theta}_2)}$$

as a measure of the relative efficiency of $\hat{\theta}_2$ w.r.t $\hat{\theta}_1$.

Example:
Suppose $X_1, X_2, \cdots, X_n$ is an i.i.d. random sample from a Poisson distribution with parameter λ. Let $\hat{\lambda}_1 = \bar{X}$ and $\hat{\lambda}_2 = \frac{X_1+X_2}{2}$ be two unbiased estimators of λ. Find the relative efficiency of $\hat{\lambda}_2$ w.r.t. $\hat{\lambda}_1$.

Consistent estimators:

Definition:

The estimator $\hat{\theta}$ of a parameter θ is said to be consistent estimator if for any positive ϵ

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta| \leq \epsilon) = 1$$

or

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \epsilon) = 0$$

We say that $\hat{\theta}$ converges in probability to θ (also known as the weak law of large numbers). In other words: the average of many independent random variables should be very close to the true mean μ with high probability.

Theorem:

An unbiased estimator $\hat{\theta}$ of a parameter θ is consistent if $var(\hat{\theta}) = 0$ as $n \to \infty$.

MSE and bias:
The bias B of an estimator $\hat{\theta}$ is given by

$$B = E(\hat{\theta}) - \theta$$

In general, given two unbiased estimators we would choose the estimator with the smaller variance. However this is not always possible (there may exist biased estimators with smaller variance). We use the mean square error (MSE)

$$MSE = E(\hat{\theta} - \theta)^2$$

as a measure of the goodness of an estimator. We can show that

$$MSE = var(\hat{\theta}) + B^2$$

Example: (From "Mathematical Statistics with Application", by Wackerly, Mendenhall, Scheaffer).
The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta+1)$, where θ is the true but unknown voltage of he circuit. Suppose that $X_1, X_2, \cdots, X_n$ denotes a random sample of such readings.

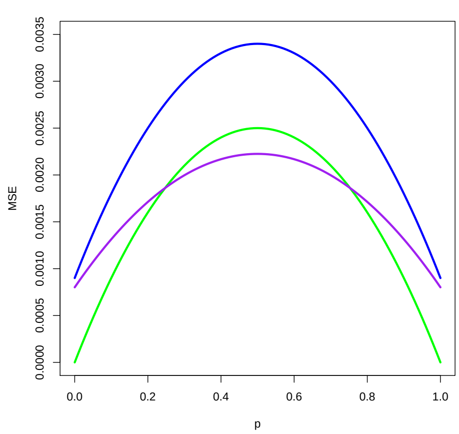
a. Show that $\bar{X}$ is a biased estimator of θ, and compute the bias.

b. Find a function of $\bar{X}$ that is an unbiased estimator of θ.

c. Find the MSE when $\bar{X}$ is used as an estimator of θ.

d. Find the MSE when the bias corrected estimator is used.

Example: (From "Theoretical Statistics", by Robert W. Keener).
Let $X \sim b(100, p)$. Consider the three estimator, $\hat{p}_1 = \frac{X}{100}, \hat{p}_2 = \frac{X+3}{100}, \hat{p}_3 = \frac{X+3}{106}$. Find the MSE for each estimator and plot it against p,



Information and Cramér-Rao inequality

Let X be a random variable with pdf $f(x;\theta)$. Then

$$\int_{-\infty}^{\infty} f(x;\theta)dx = 1 \quad \text{take derivatives w.r.t. } \theta \text{ on both sides}$$

$$\int_{-\infty}^{\infty} \frac{\partial f(x;\theta)}{\partial \theta} dx = 0 \quad \text{this is the same as:}$$

$$\int_{-\infty}^{\infty} \frac{1}{f(x;\theta)} \frac{\partial f(x;\theta)}{\partial \theta} f(x;\theta)dx = 0 \quad \text{or}$$

$$\int_{-\infty}^{\infty} \frac{\partial lnf(x;\theta)}{\partial \theta} f(x;\theta)dx = 0 \quad \text{differentiate again w.r.t. } \theta$$

$$\int_{-\infty}^{\infty} \left[\frac{\partial^2 lnf(x;\theta)}{\partial \theta^2} f(x;\theta) + \frac{\partial lnf(x;\theta)}{\partial \theta} \frac{\partial f(x;\theta)}{\partial \theta} \right] dx = 0 \quad \text{or}$$

$$\int_{-\infty}^{\infty} \left[\frac{\partial^2 lnf(x;\theta)}{\partial \theta^2} f(x;\theta) + \frac{\partial lnf(x;\theta)}{\partial \theta} \frac{1}{f(x;\theta)} \frac{\partial f(x;\theta)}{\partial \theta} f(x;\theta) \right] dx = 0 \quad \text{or}$$

$$\int_{-\infty}^{\infty} \left[\frac{\partial^2 lnf(x;\theta)}{\partial \theta^2} f(x;\theta) + \left(\frac{\partial lnf(x;\theta)}{\partial \theta} \right)^2 f(x;\theta) \right] dx = 0 \quad \text{or}$$

$$\int_{-\infty}^{\infty} \frac{\partial^2 lnf(x;\theta)}{\partial \theta^2} f(x;\theta)dx + \int_{-\infty}^{\infty} \left(\frac{\partial lnf(x;\theta)}{\partial \theta} \right)^2 f(x;\theta)dx = 0 \quad \text{or}$$

$$E\left(\frac{\partial^2 lnf(X;\theta)}{\partial \theta^2} \right) + E\left(\frac{\partial lnf(X;\theta)}{\partial \theta} \right)^2 = 0 \quad \text{or}$$

$$E\left(\frac{\partial lnf(X;\theta)}{\partial \theta} \right)^2 = -E\left(\frac{\partial^2 lnf(X;\theta)}{\partial \theta^2} \right)$$

The expression

$$E\left(\frac{\partial lnf(X;\theta)}{\partial \theta} \right)^2 = I(\theta)$$

is the so called information for one observation. The information can also be computed using the variance of the score function, $S = \frac{\partial lnf(x;\theta)}{\partial \theta}$, i.e. $I(\theta) = \text{var}\left(\frac{\partial lnf(x;\theta)}{\partial \theta} \right)$. Why?

Let's find the information in a sample: Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from a distribution with pdf $f(x;\theta)$. The joint pdf of $X_1, X_2, \cdots, X_n$ is

$$L(\theta) = f(x_1;\theta)f(x_2;\theta)\cdots\cdots f(x_n;\theta)$$

Take logarithms on both sides $\cdots$

$$lnL(\theta) = lnf(x_1;\theta) + lnf(x_2;\theta) + \cdots + lnf(x_n;\theta)$$

Take derivatives w.r.t θ on both sides

$$\frac{\partial lnL(\theta)}{\partial\theta} = \frac{\partial lnf(x_1;\theta)}{\partial\theta} + \frac{\partial lnf(x_2;\theta)}{\partial\theta} + \cdots + \frac{\partial lnf(x_n;\theta)}{\partial\theta}$$

When one observation was involved (see previous page) the information was $E\left(\frac{\partial lnf(X;\theta)}{\partial\theta}\right)^2$. Now we are dealing with a random sample $X_1, X_2, \cdots, X_n$ and $f(x;\theta)$ is replaced by $L(\theta)$ (the joint pdf). Therefore, the information in the sample will be $E\left(\frac{\partial lnL(\theta)}{\partial\theta}\right)^2$.

$$\left(\frac{\partial lnL(\theta)}{\partial\theta}\right)^2 = \left(\frac{\partial lnf(x_1;\theta)}{\partial\theta} + \frac{\partial lnf(x_2;\theta)}{\partial\theta} + \cdots + \frac{\partial lnf(x_n;\theta)}{\partial\theta}\right)^2 \quad \text{or}$$

$$\begin{aligned}\left(\frac{\partial lnL(\theta)}{\partial\theta}\right)^2 &= \left(\frac{\partial lnf(x_1;\theta)}{\partial\theta}\right)^2 + \left(\frac{\partial lnf(x_2;\theta)}{\partial\theta}\right)^2 + \cdots + \left(\frac{\partial lnf(x_n;\theta)}{\partial\theta}\right)^2 \\ &+ 2\frac{\partial lnf(x_1;\theta)}{\partial\theta}\frac{\partial lnf(x_2;\theta)}{\partial\theta} + \cdots\end{aligned}$$

Take expected values on both sides

$$E\left(\frac{\partial lnL(\theta)}{\partial\theta}\right)^2 = E\left(\frac{\partial lnf(X_1;\theta)}{\partial\theta}\right)^2 + E\left(\frac{\partial lnf(X_2;\theta)}{\partial\theta}\right)^2 + \cdots + E\left(\frac{\partial lnf(X_n;\theta)}{\partial\theta}\right)^2$$

The expected value of the cross-product terms is equal to zero. Why?

We conclude that the information in the sample is:

$$E\left(\frac{\partial lnL(\theta)}{\partial\theta}\right)^2 = I(\theta) + I(\theta) + \cdots + I(\theta)$$

or

$$I_n(\theta) = nI(\theta)$$

The information in the sample is equal to n times the information for one observation.

Cramér-Rao inequality:

$$var(\hat{\theta}) \geq \frac{1}{nI(\theta)} \quad \text{or} \quad var(\hat{\theta}) \geq \frac{1}{nE\left(\frac{\partial lnf(X;\theta)}{\partial\theta}\right)^2} \quad \text{or} \quad var(\hat{\theta}) \geq \frac{1}{-nE\left(\frac{\partial^2 lnf(X;\theta)}{\partial\theta^2}\right)}$$

Let $X_1, X_2, \cdots, X_n$ be an i.i.d. random sample from a distribution with pdf $f(x;\theta)$, and let $\hat{\theta} = g(X_1, X_2, \cdots, X_n)$ be an unbiased estimator of the unknown parameter θ. Since $\hat{\theta}$ is unbiased, it is true that $E(\hat{\theta}) = \theta$, or

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} g(x_1, x_2, \cdots, x_n) f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta) dx_1 dx_2 \cdots dx_n = \theta$$

Take derivatives w.r.t θ on both sides

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} g(x_1, x_2, \cdots, x_n) \left[\sum_{i=1}^{n} \frac{1}{f(x_i;\theta} \frac{\partial f(x_i;\theta)}{\partial\theta}\right] f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta) dx_1 dx_2 \cdots dx_n = 1$$

Since

$$\frac{1}{f(x_i;\theta)} \frac{\partial f(x_i;\theta)}{\partial\theta} = \frac{\partial lnf(x_i;\theta)}{\partial\theta}$$

we can write the previous expression as

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} g(x_1, x_2, \cdots, x_n) \left[\sum_{i=1}^{n} \frac{\partial lnf(x_i;\theta)}{\partial\theta}\right] f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta) dx_1 dx_2 \cdots dx_n = 1$$

or

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} g(x_1, x_2, \cdots, x_n) Q f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta) dx_1 dx_2 \cdots dx_n = 1$$

where

$$Q = \sum_{i=1}^{n} \frac{\partial lnf(x_i;\theta)}{\partial\theta}$$

But also, $\hat{\theta} = g(X_1, X_2, \cdots, X_n)$. So far we have $E\hat{\theta}Q = 1$. Now find the correlation between $\hat{\theta}$ and Q.