PROBLEM 1:

(a). \( E \bar{X} = E \frac{1}{n} \sum X_i = \frac{1}{n} \eta \theta = \theta \)

\( E \bar{Y} = \frac{2}{n} \sum EY_i = \frac{2}{n} \sum (EX_i)(EU_i) = \frac{2}{n} \eta \theta \frac{1}{2} = \theta \)

Both are unbiased.

\( \text{VAR}(2\bar{Y}) = 4 \text{VAR} \left( \frac{\sum Y_i}{n} \right) = \frac{4}{n} \left[ \text{VAR}(EY_i) + (EY_i)^2 \right] \quad \star \)

Note: \( EY_i^2 = E(X_i^2U_i^2) = (1+\theta^2) \left( \frac{1}{3} + \frac{1}{2} \right) = \frac{1 + \theta^2}{3} \)

\( EY_i = E(X_i)(EU_i) = E \eta U_i = \frac{2}{3} \eta \)

\( \star = \frac{4}{n} \eta \left[ \frac{1 + \theta^2}{3} - \frac{\theta^2}{4} \right] = \frac{4 + \theta^2}{3n} \)

\( \text{VAR}(\bar{X}) = \frac{1}{n} \)

Relative efficiency of \( 2\bar{Y} \) with respect to \( \bar{X} \) is:

\( \frac{\frac{4 + \theta^2}{3n}}{\frac{1}{n}} = \frac{4 + \theta^2}{3} > 1. \)
(b). Find $c$ such that $E \hat{\theta} = \theta$.

$$E \hat{\theta} = \frac{c}{\eta} \sum E(\chi_i^2)^{1/2}$$

Multiply and divide by $\sigma^2$

$$= \frac{c \sigma}{\eta} \sum \frac{E(\chi_i^2)^{1/2}}{\Gamma(\frac{k}{2})}$$

To get $\frac{\chi_i^2}{\sigma^2} \sim \chi_1^2$

or $\Gamma(\frac{k}{2}, \sigma)$

$$= \frac{c \sigma}{\eta} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \cdot 2 \cdot \frac{1}{\sqrt{\pi}} = c \sigma \sqrt{\pi}$$

Now, $E \frac{c \sigma \sqrt{\pi}}{\sqrt{\pi}} = \sigma$, therefore $c = \sqrt{\frac{\pi}{2}}$

$$\hat{\theta} = \sqrt{\frac{\pi}{2}} \frac{1}{n} \sum (\chi_i^2)^{1/2} = \sigma \sqrt{\frac{\pi}{2}} \frac{1}{n} \sum \left(\frac{X_i^2}{\sigma^2}\right)^{1/2}$$

$$\text{VAR} \left(\frac{X_i^2}{\sigma^2}\right)^{1/2} = E\left[\Gamma\left(\frac{k}{2}\right)\right] - \left(\Gamma\left(\frac{k}{2}\right)\right)^2$$

$$= \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - \left(\frac{\Gamma\left(\frac{k}{2}\right)}{\pi}\right)^2 = 3 - \frac{2}{\pi} \rightarrow \text{VAR}(\hat{\theta}) = \frac{\sigma^2}{2n} \cdot \frac{\pi(3 - \frac{2}{\pi})}{n}$$

The MLE of $\sigma$ is $\hat{\sigma} = \sqrt{\frac{1}{n} \sum X_i^2}$

The Cramer-Rao Lower Bound is $\frac{\sigma^2}{2n}$.
The median is at position \( j = n + 1 \).

Therefore,

\[
Q_{X(n+1)}(x) = \frac{(2n+1)!}{(2n+1-n-1)! (n+1-1)!} x^{n+1-1} (1-x)^{2n+1-n-1}
\]

\[
= \frac{\Gamma(2n+2)}{\Gamma(n+1) \Gamma(n+1)} x^{n+1-1} (1-x)^{n+1-1}
\]

which is the same as

\[
\frac{(n+1)-1}{B(n+1, n+1)} x^{n+1-1} (1-x)^{n+1-1}
\]

Therefore the median follows \( \text{Beta}(n+1, n+1) \).
\( (1 \theta), Y_1, \ldots, Y_n \sim U(0, \theta) \) 
\( f(y) = \frac{1}{\theta} \)

Consider \( Y_{(k)} \) and \( Y_{(k-1)} \) 
\( F(y) = \frac{y}{\theta} \)

PM PDF of \( Y_{(k)} \):

\[
\begin{align*}
\mathbb{P}(Y_{(k)}(x) = \frac{n!}{(n-k)! \cdot (k-1)!} \left( \frac{y}{\theta} \right)^{k-1} \left( 1 - \frac{y}{\theta} \right)^{n-k} \cdot \frac{1}{\theta} \\
= \frac{n!}{(n-k)! \cdot (k-1)!} \cdot \frac{y^{k-1} \theta^{-n}}{\theta} \\
\end{align*}
\]

Find Expected:

\[
\begin{align*}
E Y_{(k)} &= \frac{n!}{(n-k)! \cdot (k-1)!} \int_0^{\theta} \frac{y^k (\theta-y)^{n-k}}{\theta^n} dy \\
&= \frac{k}{n+1} \cdot \frac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} \int_0^\theta \left( \frac{y}{\theta} \right)^k \left( 1 - \frac{y}{\theta} \right)^{n-k} dy \\
\text{Let } t &= \frac{y}{\theta} \\
&= \frac{k}{n+1} \cdot \frac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} \int_0^\theta z^k (1-z)^{n-k} dz \\
&= \frac{k}{n+1} \cdot \frac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} B_{(k+1)-1, (n-k+1)-1} \\
&= \frac{k}{n+1} \cdot \frac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} B(2, n) \\
\alpha &= k+1 \\
\beta &= n-k+1
\end{align*}
\]
Similarly, \( E Y_{(k-1)} = \frac{k - 1}{n+1} \theta \)

Therefore,

\[ E Y_k - E Y_{(k-1)} = \frac{k}{n+1} \theta - \frac{k - 1}{n+1} \theta = \frac{1}{n+1} \theta. \]
PROBLEM 2:

(a) Joint PMF of $X_i$'s since they are independent is:

\[
P(X_1, \ldots, X_n) = \frac{2^n e^{-n}}{x_1! \cdot x_2! \cdots x_n!}
\]

\[
P(S) = \frac{(nN)^N e^{-nN}}{N!}, \text{ because } \sum X_i \sim \text{Poisson}(nN)
\]

Therefore,

\[
P\left(X_1 = x_1, \ldots, X_n = x_n \mid S = N\right) = \frac{P\left(\left(X_1 = x_1, \ldots, X_n = x_n\right) \cap (S = N)\right)}{P(S = N)}
\]

Note: \(P\left(\left(X_1 = x_1, \ldots, X_n = x_n\right) \cap (S = N)\right) = \frac{2^n e^{-nN}}{x_1! \cdot \ldots \cdot x_n!}
\]

\[
= \frac{2^n e^{-nN}}{(nN)^N e^{-nN}} = \frac{N!}{n^N x_1! \cdot \ldots \cdot x_n!}
\]

\[
= \frac{N!}{x_1! \cdot \ldots \cdot x_n!} \left(\frac{1}{n}\right)^{x_1} \cdot \left(\frac{1}{n}\right)^{x_2} \ldots \left(\frac{1}{n}\right)^{x_n}
\]

where \(\sum X_i = N\).
(b). \( E\left(\frac{X}{X-1}\right) = \lambda \), \( \text{VAR}\left(\frac{X}{X-1}\right) = 2\lambda^2 + 4\lambda^3 \)

Let \( T_i = \frac{1}{n} \sum X_i (X_i - 1) \)

\[ E(T_i) = \frac{1}{n} \sum E(X_i (X_i - 1)) = \frac{1}{n} n \lambda^2 = \lambda^2 \]

\[ \text{VAR}(T_i) = \frac{1}{n} \sum \text{VAR}(X_i (X_i - 1)) \]

\[ = \frac{1}{n^2} \left( 2\lambda^2 + 4\lambda^3 \right) = \frac{2\lambda^2 + 4\lambda^3}{n} \]

To verify mean and variance of \( X (X-1) \):

\[ E(X (X-1)) = E(X^2 - X) = EX^2 - EX \]

\[ = (\sigma^2 + \lambda) - \lambda = \lambda + \lambda^2 - \lambda = \lambda \]

\[ \text{AND} \]

\[ \text{VAR}(X (X-1)) = \text{VAR}(X^2) + \text{VAR}(X) - 2 \text{COV}(X^2, X) \]

\[ = E X^4 - (EX^2)^2 + \text{VAR}(X) - 2\left[ E X^3 -(EX^2)(EX) \right] \]

ETC.
(2). \( T_2 = E[T_1 \mid S] \), where \( S = \sum X_i \).

It is given that \( E[X_i \mid S] = \frac{S}{n} \)

\[
\text{Var} \left[ X_i \mid S \right] = S \frac{1}{n} \left( 1 - \frac{1}{n} \right) \rightarrow E \frac{X_i^2}{S} = \frac{S + (n-1)S}{n^2}
\]

Therefore,

\[
T_2 = E \left[ T_1 \mid S \right] = E \left[ \frac{1}{n} \sum X_i (X_i - 1) \mid S \right]
\]

\[
= E \left[ \frac{1}{n} \sum X_i^2 - \frac{1}{n} \sum X_i \mid S \right]
\]

\[
= E \left[ \frac{1}{n} \sum X_i^2 \mid S \right] - E \left[ \frac{1}{n} \sum X_i \mid S \right]
\]

\[
= \frac{1}{n} \cdot \frac{n \left( S + (n-1)S \right)}{n^2} - \frac{1}{n} \cdot \frac{n \cdot S}{n} = \frac{S^2 + (n-1)S}{n^2} - \frac{S}{n}
\]

\[
= \frac{S (S - 1)}{n}.
\]

But \( S \sim \text{Exp}(\alpha) \)

Therefore \( E T_2 = E \frac{S (S - 1)}{n} = \frac{E S^2 - ES}{n} = \frac{n^2 \alpha + n \alpha - \alpha}{n^2} \)

\[
E S = \frac{n \alpha}{n - 1}
\]

Therefore \( E S = \frac{n \alpha}{n - 1} \)

Variance of \( T_2 \): From (1)

\[
\text{Var}[X (X-1)] = 2 \alpha^2 + 4 \alpha^3
\]

Here \( \text{Var}[S (S-1)] = 2 \left( \frac{n \alpha^2}{n - 1} \right) + 4 \left( \frac{n \alpha^3}{n - 1} \right) \), to get

\[
\text{Var} \left( T_2 \right) = \frac{2 \alpha^2}{n} + \frac{4 \alpha^3}{n}.
\]
\( (d). \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\theta} \hat{Q} \, dx_1 \cdots dx_n = \lambda \)

\[
E \hat{\theta} \hat{Q} = 2 \Lambda
\]

\[
P(\hat{\theta} = 1) \quad \text{or} \quad \frac{\text{cov}(\hat{\theta}, \hat{Q})}{\text{var}(\hat{\theta}) \text{var}(\hat{Q})} \neq 1
\]

**Note:** \( E Q = 0 \), \( \text{var}(\hat{Q}) = \frac{n}{\lambda} \)

\[
\left( E \hat{\theta} \hat{Q} - (E \hat{\theta})(E \hat{Q}) \right)^2 \leq 1
\]

\[
\frac{4 \lambda^2}{\text{var}(\hat{\theta}) n \lambda} \leq 1 \quad \Rightarrow \quad \text{var}(\hat{\theta}) > \frac{4 \lambda^2}{n \lambda}
\]

Now, find \( I(\theta) \):

\[
P(x) = \frac{e^{-x^2/2}}{x} \quad \Rightarrow \quad p(x) = x \text{d}e^{x^2/2} = x - 1
\]

\[
\frac{\partial}{\partial \theta} p(x) = \frac{-1}{2} \quad \Rightarrow \quad \therefore \quad I(\theta) = \frac{1}{2}
\]

Finally, \( \text{var}(\hat{\theta}) = \frac{4 \lambda^3}{n} \)
$T_1$ and $T_2$ are unbiased estimators of $\theta$.

We found in (b) that

$$\text{VAR}(T_1) = \frac{2\theta^2 + 4\theta^3}{n}$$

And in (c) $\text{VAR}(T_2) = \frac{2\theta^2 + 4\theta^3}{n^2}$

Therefore they do not attain the Cramer–Rao lower bound.

$T_2$ has variance closer to the Cramer–Rao lower bound.
**Problem 3:**

(a) \( y_0 = \beta_0 + b_i \bar{x} \)

\[ y_i = y_0 + b_i z_i + \epsilon_i \quad \text{where} \quad z_i = x_i - \bar{x} \]

\[ y_i \sim N \left( y_0 + b_i z_i, \sigma \right) \]

\[ L = \left( \frac{2 \pi \sigma}{2} \right)^{-\frac{1}{2}} e^{-\frac{1}{2 \sigma^2} \sum (y_i - y_0 - b_i z_i)^2} \]

\[ \log L = -\frac{n}{2} \log 2 \pi \sigma - \frac{1}{2 \sigma^2} \sum (y_i - y_0 - b_i z_i)^2 \]

\[ \frac{\partial \log L}{\partial b_1} = \frac{1}{\sigma^2} \sum (y_i - y_0 - b_1 z_i) z_i = 0 \]

\[ \frac{\partial \log L}{\partial \beta_0} = \frac{1}{\sigma^2} \sum (y_i - y_0 - b_1 z_i) = 0 \]

\[ \hat{\beta}_1 = \frac{\sum (z_i - \bar{z})(y_i - \bar{y})}{\sum (z_i - \bar{z})^2} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \]

(Note: \( \bar{z} = 0 \))

And \( \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \bar{y} \).
\( (b), \hat{\theta}_1 \sim N(\theta_1, \frac{\sigma}{\sqrt{\sum (X - \bar{X})^2}}) \)

\( \hat{\sigma}_0 \sim N(\sigma_0, \frac{\sigma}{\sqrt{n}}) \)

Joint PDF is BIVARIATE NORMAL

with \( \text{cov}(\bar{Y}, \hat{\theta}_1) = 0 \)

Therefore, \( \bar{Y}, \hat{\theta}_1 \) are independent.

**Estimate of \( \sigma^2 \):**

\[
\frac{dL}{d\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (y_i - \hat{\delta}_0 - \hat{\theta}_1 (x_i - \bar{x}))^2 = 0
\]

\[
\hat{\sigma}^2 = \frac{\sum (y_i - \hat{\delta}_0 - \hat{\theta}_1 (x_i - \bar{x}))^2}{n}
\]
\[
\sum_{i=0}^{n} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x}) \right)^2
\]

\[
= \frac{1}{\sigma^2} \sum_{i=0}^{n} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x}) \right)^2 + \frac{2 \hat{\beta}_1}{\sigma^2} \sum_{i=0}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x})) + \frac{1}{\sigma^2} \sum_{i=0}^{n} \sum_{j=0}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x})) (y_j - \hat{\beta}_0 - \hat{\beta}_1 (x_j - \bar{x}))
\]

\[
= \frac{1}{\sigma^2} \sum_{i=0}^{n} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x}) \right)^2 + \frac{n \hat{\beta}_1^2}{\sigma^2} \sum_{i=0}^{n} (x_i - \bar{x})^2 + \frac{2 \hat{\beta}_1}{\sigma^2} \sum_{i=0}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x})) + \frac{1}{\sigma^2} \sum_{i=0}^{n} \sum_{j=0}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x})) (y_j - \hat{\beta}_0 - \hat{\beta}_1 (x_j - \bar{x}))
\]

\[
\sum_{i=0}^{n} e_i = 0
\]

\[
\sum_{i=0}^{n} x_i e_i = 0
\]

**Therefore**

\[
\sum_{i=0}^{n} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x}) \right)^2 = \frac{(n-2)s^2}{\sigma^2} + \frac{(\hat{\beta}_0 - \bar{y})^2}{\sigma^2 / \sum (x_i - \bar{x})^2} + \frac{(\hat{\beta}_1 - \bar{b})^2}{\sigma^2 / \sum (x_i - \bar{x})^2}
\]

\[
\hat{x}_0 \hat{x}_1
\]

\[
\hat{x}_n \because \overline{\sum e_i} = \hat{\beta}_0, \hat{\beta}_1
\]

\[
\hat{x}_n \because \text{because } \sum e_i = \hat{\beta}_0, \hat{\beta}_1
\]

\[
\text{are independent.}
\]
(d) \( e_i \sim N(0, \sigma \sqrt{1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2}}) \)

\[
\text{and} \quad \frac{(n-2)\hat{e}}{\sigma} \sim \chi^2_{n-2}.
\]

However, \( e_i \) and \( \hat{e} = \sum e_i^2 / (n-2) \) are not independent, therefore the ratio does not follow the \( t \) distribution.
Problem 4:

(a) \( Y_1, \ldots, Y_n \sim iid \, N(\mu, \sigma^2) \)

\[ \sigma = \frac{Y'(I - \frac{1}{n}11')Y}{n} \]

\[ E \, Y'(I - \frac{1}{n}11')Y = E \, tr \, Y'(I - \frac{1}{n}11')Y \]

\[ = \sigma \, (I - \frac{1}{n}11') \, E \, YY' \]

\[ = \sigma \, (I - \frac{1}{n}11') \left( \text{var}(Y) + \frac{1}{n}II \right) \]

\[ = \sigma \, (I - \frac{1}{n}11') + \sigma \]

\[ = \sigma [n - 1] \]

\[ \therefore \frac{E \, Y'(I - \frac{1}{n}11')}{\sigma} = n - 1 \]

\[ \therefore \frac{1}{\sigma} = \frac{1}{n} \, tr \, II = 1. \]

Note: \( \frac{1}{\sigma} = \frac{1}{n} \, tr \, II = 1. \)
(b) \( x_1, \ldots, x_m \sim \exp(\lambda x) \)

\( x_1, \ldots, y_m \sim \exp(\lambda y) \)

\( M_{x_i}(t) = (1 - \frac{t}{\lambda x})^{-1}, \quad M_{y_i}(t) = (1 - \frac{t}{\lambda y})^{-1} \)

\( \Sigma X_i \sim \Gamma(m, \frac{1}{\lambda x}) \), \( \Sigma Y_i \sim \Gamma(m, \frac{1}{\lambda y}) \)

\( \Sigma X \sim \Gamma(m, \frac{1}{\lambda x}) \)

Let \( \hat{\theta} = \frac{\Sigma Y_i}{\Sigma X_i} \)

Find \( E \hat{\theta} \).

\( E \hat{\theta} = (E \Sigma Y_i) \cdot (E(\Sigma X_i)^{-1}) \)

\[
= \frac{n}{\lambda y} \cdot \frac{\Gamma(m-1) \left( \frac{1}{2x} \right)^{-1}}{\Gamma(m)} = \frac{\lambda x}{\lambda y} \cdot \frac{n}{m-1}
\]

Therefore, unbiased estimator of \( \frac{dx}{dy} \)

is \( \hat{\theta} = \frac{m-1}{n} \frac{\Sigma Y_i}{\Sigma X_i} \).
We want to minimize
\[ \text{MSE} = E \left[ \delta - \frac{\partial}{\partial \gamma} \right]^2 = E \left[ w \frac{\Sigma y_i}{\Sigma x_i} - \frac{\partial}{\partial \gamma} \right]^2 = \lambda. \]

Note: \( \frac{\Sigma y_i}{\Sigma x_i} = w \frac{\Sigma x_i}{\Sigma x_i} \), where \( w = \frac{m}{m-1} \).

We need \( E(\Sigma y_i) \) and \( E(\Sigma y_i)^{-2} \).

\( E(\Sigma y_i) = \text{var}(\Sigma y_i) + (E \Sigma y_i)^{-1} = \frac{n^2 + \Sigma y_i^2}{n} = \frac{n(n+1)}{\Sigma y_i} \)

\( E(\Sigma x_i) = \frac{\Gamma(m-2)}{\Gamma(m)} \left( \frac{1}{\Sigma y_i} \right)^{\frac{1}{2}} \)

\( = \frac{\Gamma(m-2)}{(m-1)(m-2)} \frac{1}{\Gamma(m-2)} = \frac{dx}{(m-1)(m-2)} \)

\[ x = w^2 E \frac{(\Sigma y_i)^2}{(\Sigma x_i)} - 2w \frac{dx}{\Sigma y_i} E \frac{\Sigma y_i}{\Sigma x_i} + \frac{dx}{\Sigma y_i} \]

\[ = w^2 \frac{dx}{\Sigma y_i} \frac{n(n+1)}{m(m-1)(m-2)} - 2w \frac{dx}{\Sigma y_i} \frac{n}{m} + \frac{dx}{\Sigma y_i} \]

This minimized when \( w = \frac{m-2}{n+1} \)

And therefore \( \delta = \frac{n(m-2)}{m(n+1)} \)
(d) $X \sim \exp(\theta_x), \ Y \sim \exp(\theta_y)$

Given $\theta_x = 2\theta_y$

$L = \theta_x e^{-\theta_x X} \cdot \theta_y e^{-\theta_y Y} = 2\theta_y e^{-\theta_y (X+2Y)}$

$L = 2\theta_y e^{-\theta_y (X+2Y)}$

$$\ln L = \ln 2\theta_y - \theta_y (X+2Y)$$

$$\frac{d\ln L}{d\theta_y} = \frac{2\theta_y}{\theta_y} - (X+2Y) = 0$$

$$\hat{\theta}_y = \frac{2}{X+2Y}$$

$X+Y$ does not follow gamma distribution.
\( E(Y_i) = \mu, \quad \text{var}(Y_i) = \sigma^2 \)

\begin{align*}
\text{cov}(Y_i, Y_j) &= \rho \sigma^2 \\
E(Y) &= E\left( \frac{\sum Y_i}{n} \right) = \frac{\mu}{n} \\
\text{var}(Y) &= \text{var}\left( \frac{\sum Y_i}{n} \right) = \frac{1}{n^2} \text{var}(\sum Y_i) \\
&= \frac{\sigma^2}{n} \left[ 1 + \left( n-1 \right) \rho \right] \\
&\xrightarrow{n \to \infty} \rho \sigma^2
\end{align*}

As \( n \to \infty \)

**Therefore** \( \bar{Y} \) is **not** a consistent estimator of \( \mu \).