University of California, Los Angeles Department of Statistics

Statistics 100B

Instructor: Nicolas Christou

Final exam - practice problems

Problem 1

Answer the following questions:

- a. A merchant figures his weekly profit to be a function of three variables: X, Y, Z (retail sales, wholesale sales, overhead costs). The variables X, Y, Z are independent, normally distributed random variables with means μ_1, μ_2, μ_3 , and variances $\sigma^2, a\sigma^2, b\sigma^2$, respectively, for known constants a and b but unknown σ^2 . The merchant's expected profit per week is $\mu_1 + \mu_2 \mu_3$. If the merchant's has independent observations for X, Y, Z for the past n weeks, construct a test of $H_0: \mu_1 + \mu_2 \mu_3 = k$ against the alternative $H_a: \mu_1 + \mu_2 \mu_3 \neq k$.
- b. Let X_1, X_2, \ldots, X_n be independent, let $f(x_i; \theta) = \theta x_i^{\theta 1}$. Find the form of the most powerful test for testing $H_0: \theta = 1$

$$H_a: \theta = 2$$

at the level of significance α .

- c. Suppose that four observations are taken at random from a normal distribution for which the mean μ is unknown and the variance is 1. The following hypothesis will be tested:
 - $H_0: \mu \ge 10$ $H_a: \mu < 10$
 - 1. Determine a uniformly most powerful test at the level of significance $\alpha = 0.10$.
 - 2. Determine the power of this test when $\mu = 9$.
- d. Let $X \sim N(\mu, \sigma)$ (both μ and σ^2 are unknown). We want to test $H_0: \mu = 1.80$ against $H_a: \mu > 1.80$ using $\alpha = 0.10$ level of significance. If a random sample of size n = 121 gave $\bar{x} = 1.84$ and s = 0.20, is H_0 accepted? Also, use your t table to approximate the p-value for this test.

Problem 2

Answer the following questions:

- a. Let X_1, X_2, \dots, X_9 and Y_1, Y_2, \dots, Y_{12} represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = 3\sigma_2^2$, but σ_2^2 is unknown. Define a random variable which has a t distribution and use it to find a 95% confidence interval for $\mu_1 \mu_2$.
- b. Let X_1, X_2, \dots, X_{100} be a random sample of size n = 100 from a Poisson distribution with parameter λ . We will reject $H_0: \lambda = 4$ and accept $H_a: \lambda > 4$ if the observed sum is $\sum_{i=1}^{100} X_i > 440$. Write the expression that computes the exact Type II error probability β if the true parameter is $\lambda = 4.7$, and then approximate it.
- c. The difference of the means of two normal distributions with equal and known variances is to be estimated by sampling an equal number of observations from each distribution. If it were possible, would it be better (in terms of the width of the resulting confidence interval) (a) to halve the standard deviations of the populations or (b) to double the sample size for each sample? For both confidence intervals use 95% confidence level. Explain your answer.
- d. Let X_1, X_2, X_3 be a random sample of size n = 3 from an exponential distribution with mean θ . The null hypothesis $H_0: \theta = 2$ will be rejected (and the alternative $H_a: \theta < 2$ will be accepted) if the observed sum $\sum_{i=1}^{3} x_i \leq 2$. Find the Type I error α . If $\theta = 0.35$ find the Type II error β .

Problem 3

Let X_1, X_2, \ldots, X_n be a random sample from $N(0, \sigma)$.

a. Show that $\sum_{i=1}^n X_i^2 > k'$ is the best critical region for testing $H_0:\sigma^2=4$

$$H_a:\sigma^2=16$$

using the Neyman-Pearson lemma.

- b. If n = 15, find the value of k' so that $\alpha = 0.05$.
- c. If n = 15 and k' is the value found in part (b), compute

$$\beta = P(\sum_{i=1}^n X_i^2 < k' \ \, {\rm when} \ \, \sigma^2 = 16).$$

Problem 4

Consider the ANOVA problem with k = 2 groups. Show that the ANOVA F statistic for testing the hypothesis $H_0: \mu_1 = \mu_2$ against $H_a: \mu_1 \neq \mu_2$ is equivalent to the sample t test when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (but unknown).

Problem 5

Let Y_1, Y_2, \ldots, Y_n denote a random sample from a Poisson distribution with mean λ_1 and let X_1, X_2, \ldots, X_m denote a random sample from a Poisson distribution with mean λ_2 . The two samples are independent.

a. Derive the most powerful test for testing $H_0: \lambda_1 = 2, \lambda_2 = 2$

 $H_a: \lambda_1 = 3, \lambda_2 = 3.$

b. Let n = 5 and m = 10. Find the rejection region if $\alpha = 0.05$. The table below gives $P(X \le x)$ for different values of x and λ .

x	$\lambda = 25$	$\lambda = 30$	$\lambda = 35$
30	0.86331	0.54835	0.22694
31	0.89993	0.61864	0.28328
32	0.92854	0.68454	0.34489
33	0.95022	0.74445	0.41025
34	0.96616	0.79731	0.47752
35	0.97754	0.84262	0.54479
36	0.98545	0.88037	0.61020
37	0.99079	0.91099	0.67207
38	0.99430	0.93516	0.72905
39	0.99656	0.95375	0.78019
40	0.99796	0.96769	0.82494

Problem 6

To test a hypothesis about the equality of two parameters λ_1 and λ_2 we use two independent samples X_1, X_2, \ldots, X_m and Y_1, Y_2, \ldots, Y_n selected from two exponential distributions with parameters λ_1 and λ_2 and the following rejection region was found: R.R. = { $\frac{\bar{X}}{\bar{V}} > k$ }. Suggest an exact test based on the distribution of this test statistic.

Problem 7

A reading exam is given to sixth grades at three large elementary schools. The scores on the exam at each school are regarded as having normal distributions with unknown means μ_1, μ_2, μ_3 respectively and unknown variance $\sigma^2(\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2)$. Using the data in the table below test to see if there is evidence of a difference between μ_1 and μ_2 . Use $\alpha = 0.05$.

School I	School II	School III
$n_1 = 10$ $\sum r_1^2 = 36950$	$n_2 = 10$ $\sum u_1^2 - 25850$	$n_3 = 10$ $\sum w_1^2 - 49900$
$\sum x_i = 50550$ $\bar{x} = 60$	$\sum_{i} \frac{g_i}{\bar{y}} = 50$	$\sum w_i = 10000$ $\bar{w} = 70$

Problem 8

Let X_1, \ldots, X_n be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both \bar{X} and cS are unbiased estimators of θ , where $c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$. For what value of α the estimator $\hat{\theta} = \alpha \bar{X} + (1-\alpha)cS$ has the minimum variance? Define another estimator by $T = \alpha_1 \bar{X} + \alpha_2(cS)$, where $\alpha_1 + \alpha_2 \neq 1$. Find the estimator that minimizes $E(T-\theta)^2$.