EXERCISE 1
A coin is thrown independently 10 times to test that the probability of heads is \( \frac{1}{2} \) against the alternative that the probability is not \( \frac{1}{2} \). The test rejects \( H_0 \) if either 0 or 10 heads are observed.

a. What is the significance level \( \alpha \) of the test?

b. If in fact the probability of heads is 0.1, what is the power of the test?

EXERCISE 2
The output voltage of a certain electric circuit is specified to be 130 volts. The population standard deviation is known to be \( \sigma = 3.0 \) volts. A sample of 40 readings on the voltage of this circuit gave a sample mean of 128.6 volts.

a. Test the hypothesis that the mean output voltage is 130 volts against the alternative that it is less than 130 volts. Use \( \alpha = 0.05 \).

b. Based on your answer to (a), is it possible that the mean output voltage is still 130 volts? Explain.

c. If the true population mean output voltage is 128.6 volts, compute the probability of a type II error (\( \beta \)) and the power of the test (1 - \( \beta \)) when \( \alpha = 0.05 \).

d. For this part you do not have to show any calculations.

   i. The type I error \( \alpha \) decreases to 0.01?

   ii. The true population mean is 129.6 volts?

EXERCISE 3
Answer the following questions:

a. The lifetime of certain batteries are supposed to have a variance of 150 hours\(^2\). Using \( \alpha = 0.05 \) test the following hypothesis

\[ H_0 : \sigma^2 = 150 \]
\[ H_a : \sigma^2 > 150 \]

if the lifetimes of 15 of these batteries (which constitutes a random sample from a normal population) have:

\[ \sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000. \]

where \( X \) denotes the lifetime of a battery.

b. A confidence interval is unbiased if the expected value of the interval midpoint is equal to the estimated parameter. For example the midpoint of the interval \( \bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \) is \( \bar{x} \), and \( E(\bar{x}) = \mu \). Now consider the confidence interval for \( \sigma^2 \). Show that the expected value of the midpoint of this confidence interval is not equal to \( \sigma^2 \).

EXERCISE 4
Let \( X \) be a uniform random variable on \((0, \theta)\). You have exactly one observation from this distribution and you want to test the null hypothesis \( H_0 : \theta = 10 \) against the alternative \( H_a : \theta > 10 \), and you want to use significance level \( \alpha = 0.10 \). Two testing procedures are being considered:

Procedure \( G \) rejects \( H_0 \) if and only if \( X \geq 9.5 \).

Procedure \( K \) rejects \( H_0 \) if either \( X \geq 9.5 \) or if \( X \leq 0.5 \).

a. Confirm that Procedure \( G \) has a Type I error probability of 0.10.

b. Confirm that Procedure \( K \) has a Type I error probability of 0.10.

c. Find the power of Procedure \( G \) when \( \theta = 12 \).

d. Find the power of Procedure \( K \) when \( \theta = 12 \).
EXERCISE 5
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter $\lambda$. Find the best critical region for testing
   $H_0 : \lambda = 2$
   $H_a : \lambda = 5$
   using the Neyman-Pearson lemma.

b. Let $Y_1, Y_2, \ldots, Y_n$ be the outcomes of $n$ independent Bernoulli trials. Find the best critical region for testing
   $H_0 : p = p_0$
   $H_a : p > p_0$
   using the Neyman-Pearson lemma.

EXERCISE 6
Consider the simple regression model through the origin, $y_i = \beta_1 x_i + \epsilon_i$, with $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ independent, and $\sigma^2$ is known. Use the Neyman-Pearson lemma to find the most powerful test for testing
   $H_0 : \beta_1 = 0$
   $H_a : \beta_1 > 0$
at the level of significance $\alpha$ (for a fixed value of $\beta_1$ under $H_a$).

EXERCISE 7
Consider the regression model $y_i = \theta x_i^2 + \epsilon_i$, for $i = 1, \ldots, n$, with $E(\epsilon_i) = 0$, $\text{var}(\epsilon_i) = \sigma^2$, $\text{cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$ because $\epsilon_i, \epsilon_j$ are independent, and $\epsilon_i \sim N(0, \sigma)$. Show that the maximum likelihood estimate of $\theta$ is $\hat{\theta} = \frac{2 \sum_{i=1}^{n} x_i^2 y_i}{\sum_{i=1}^{n} x_i^4}$. Find the distribution of $\hat{\theta}$. Suppose $\sigma^2$ is known. Using the distribution of $\hat{\theta}$ find $1 - \alpha$ confidence interval for $\theta$. 