Exercise 1
Suppose $X$ has the possible values 0,1,2,3,4. Suppose that the null hypothesis says that $X$ is uniform on these integers, while the alternative hypothesis says that $X \sim b(4, \frac{1}{2})$. Let’s see what happens if we let $k$ of the Neyman-Pearson lemma be equal to 0.6. Complete the next table and find the best critical region when $k = 0.6$ and compute the power of the test.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x</td>
<td>H_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P(X = x</td>
<td>H_a)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exercise 2
Let $X$ be a random variable whose probability mass function under $H_0$ and $H_a$ is given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x</td>
<td>H_0)$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.94</td>
</tr>
<tr>
<td>$f(x</td>
<td>H_a)$</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Use the Neyman-Pearson lemma to find the most powerful test for testing $H_0$ against $H_a$ with $\alpha = 0.04$. Compute the probability of Type II error for this test.

Exercise 3
It is known that the random variable $X$ has a probability density function of the form $f(x) = \frac{1}{\theta} e^{-x/\theta}$. We want to test the hypothesis $H_0 : \theta = 2$ against $H_a : \theta = 4$. A random sample $X_1, X_2$ of size $n = 2$ will be selected. Suppose the rejection region is given by $X_1 + X_2 > 9.49$. Calculate the Type I and Type II error probabilities. For the Type I error probability you should use your $\chi^2$ table. For the Type II error probability you may find the following table useful. Please show all your work.

<table>
<thead>
<tr>
<th>$\chi^2$</th>
<th>$P(\chi^2 &lt; 2 \times 9.49)$</th>
<th>$P(\chi^2 &lt; 9.49)$</th>
<th>$P(\chi^2 &lt; \frac{9.49}{2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2_2$</td>
<td>0.9913</td>
<td>0.9068</td>
<td>0.9999</td>
</tr>
<tr>
<td>$\chi^2_4$</td>
<td>0.9500</td>
<td>0.6855</td>
<td>0.9992</td>
</tr>
<tr>
<td>$\chi^2_8$</td>
<td>0.6973</td>
<td>0.2155</td>
<td>0.9850</td>
</tr>
</tbody>
</table>

Exercise 4
A coin is thrown independently 10 times to test that the probability of heads is $\frac{1}{2}$ against the alternative that the probability is not $\frac{1}{2}$. The test rejects $H_0$ if either 0 or 10 heads are observed.

a. What is the significance level $\alpha$ of the test?

b. If in fact the probability of heads is 0.1, what is the power of the test?
Exercise 5
Let \( X \) be a uniform random variable on \((0, \theta)\). You have exactly one observation from this distribution and you want to test the null hypothesis \( H_0 : \theta = 10 \) against the alternative \( H_a : \theta > 10 \), and you want to use significance level \( \alpha = 0.10 \). Two testing procedures are being considered:

Procedure \( G \) rejects \( H_0 \) if and only if \( X \geq 9 \).
Procedure \( K \) rejects \( H_0 \) if either \( X \geq 9.5 \) or if \( X \leq 0.5 \).

a. Confirm that Procedure \( G \) has a Type I error probability of 0.10.
b. Confirm that Procedure \( K \) has a Type I error probability of 0.10.
c. Find the power of Procedure \( G \) when \( \theta = 12 \).
d. Find the power of Procedure \( K \) when \( \theta = 12 \).

Exercise 6
Answer the following questions:

a. Let \( X_1, X_2, \ldots, X_n \) denote a random sample from a Poisson distribution with parameter \( \lambda \). Find the best critical region for testing

\[ H_0 : \lambda = 2 \]
\[ H_a : \lambda = 5 \]

using the Neyman-Pearson lemma.

b. Let \( Y_1, Y_2, \ldots, Y_n \) be the outcomes of \( n \) independent Bernoulli trials. Find the best critical region for testing

\[ H_0 : p = p_0 \]
\[ H_a : p > p_0 \]

using the Neyman-Pearson lemma.

Exercise 7
Let \( X_1, \ldots, X_n \) be a random sample from an exponential distribution with parameter \( \theta \). Answer the following questions: Derive a likelihood ratio test for testing \( H_0 : \theta = \theta_0 \) against \( H_a : \theta \neq \theta_0 \), and show that the rejection region is of the form \( \{ \bar{X} \exp(-\theta_0 \bar{X}) \} < c \). Suppose \( \alpha = 0.05 \) and \( \theta_0 = 1 \). Explain why \( c \) should be chosen so that \( P(\bar{X} \exp(-\bar{X}) < c) = 0.05 \).

Exercise 8
Suppose \( X_1, \ldots, X_n \) are i.i.d. Poisson(\( \lambda_1 \)) and \( Y_1, \ldots, Y_n \) are i.i.d. Poisson(\( \lambda_2 \)). The two samples are independent. Find the most powerful test for testing \( H_0 : \lambda_1 = \lambda_2 = 2 \) against \( H_a : \lambda_1 = \frac{1}{2}, \lambda_2 = 3 \).