Answer the following questions:

a. A general technique for reducing bias in an estimator is the following. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, and let $\hat{\theta}$ be some estimator of a parameter $\theta$. In order to reduce the bias the method works as follows: We calculate $\hat{\theta}^{(i)}$, $i = 1, 2, \ldots, n$, just as $\hat{\theta}$ is calculated but using the $n-1$ observations with $X_i$ removed from the sample. This new estimator is given by $\hat{\theta}^\ast = n\hat{\theta} - \frac{n-1}{n} \sum_{i=1}^{n} \hat{\theta}^{(i)}$. To apply this concept we will use the Bernoulli distribution. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli$(p)$. It is given that the MLE of $p^2$ is $\hat{\theta} = \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2$. Show that $\hat{\theta}$ is not unbiased for $p^2$.

b. Refer to question (a). Use the technique described above to reduce the bias in $\hat{\theta}$. Does the method remove the bias entirely in this example?

c. Find the Rao-Cramér lower bound of an estimator of $\theta$ but do not assume that $\hat{\theta}$ is unbiased estimator of $\theta$. Please show the entire derivation.

d. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y|\theta) = \left(\frac{2y}{\theta}\right)exp(-\frac{y^2}{\theta}), y > 0$ (Weibull distribution). Show that $\hat{\theta} = \frac{\sum_{i=1}^{n} Y_i^2}{n}$ is unbiased estimator of $\theta$. Is $\hat{\theta}$ an efficient estimator of $\theta$?

e. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta)$, $\theta > 0$. For this model both $\bar{X}$ and $cS$ are unbiased estimators of $\theta$, where $c = \frac{\sqrt{n-1}t(\frac{n-1}{2})}{\sqrt{2}t(\frac{1}{2})}$. Define the estimator $T = \alpha_1 \bar{X} + \alpha_2 (cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T - \theta)^2$.

f. Consider the following two independent sets of random variables: Let $X_1, \ldots, X_n$ i.i.d. random variables with $X_i \sim N(\mu_1, \sigma)$ and let $Y_1, \ldots, Y_m$ i.i.d. random variables with $Y_i \sim N(\mu_2, \sigma)$. Consider the random variable $S_p^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2}{n+m-2}$. Find $ES_p$. 