University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 1

EXERCISE 1

A pdf or pmf is called an exponential family if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right).$$

Express $X \sim \Gamma(\alpha, \beta)$ in this form.

EXERCISE 2

Let $X \sim \Gamma(\alpha, \beta)$. Show that for $k > 0$, $EX^k = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$.

EXERCISE 3

The two theorems we discussed in class are:

$$E\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x)\right) = -\frac{\partial}{\partial \theta_j} logc(\boldsymbol{\theta}).$$

and

$$var\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x)\right) = -\frac{\partial^2}{\partial \theta_j^2} logc(\boldsymbol{\theta}) - E\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(x)\right).$$

Note: Here log is the natural logarithm.

In class we showed that the binomial pmf can be expressed in the exponential family form and then we found $E(X) = np$ using the first theorem. Use the second theorem to show that $var(X) = np(1-p)$.

EXERCISE 4

Prove theorem 1. For the first statement of the theorem use the following:

$$\begin{aligned} \int_x f(x|\boldsymbol{\theta})dx &= 1 \\ \int_x h(x)c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right) &= 1. \end{aligned}$$

Hint: Differentiate both sides w.r.t. θ_j and rearrange to prove the first statement of the theorem.

For the second statement of the theorem differentiate a second time and rearrange.

EXERCISE 5

The probability density function of the beta distribution is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad \alpha > 0,\ \beta > 0,\ 0 < x < 1.$$

where,

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Show that $EX^n = \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$.

EXERCISE 6

Show that

1. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.
2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.