EXERCISE 1
A pdf or pmf is called an exponential family if it can be expressed as

\[ f(x|\theta) = h(x)c(\theta)exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x)\right). \]

Express \( X \sim \Gamma(\alpha, \beta) \) in this form.

EXERCISE 2
Let \( X \sim \Gamma(\alpha, \beta) \). Show that for \( k > 0 \),

\[ EX^k = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}. \]

EXERCISE 3
The two theorems we discussed in class are:

\[ E\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(x)\right) = -\frac{\partial}{\partial \theta_j} \log c(\theta). \]

and

\[ \text{var}\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(x)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) - E\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(x)\right). \]

Note: Here \( \log \) is the natural logarithm.

In class we showed that the binomial pmf can be expressed in the exponential family form and then we found \( E(X) = np \) using the first theorem. Use the second theorem to show that \( \text{var}(X) = np(1-p) \).

EXERCISE 4
Prove theorem 1. For the first statement of the theorem use the following:

\[ \int_{x} f(x|\theta)dx = 1 \]

\[ \int_{x} h(x)c(\theta)exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x)\right) = 1. \]

Hint: Differentiate both sides w.r.t. \( \theta_j \) and rearrange to prove the first statement of the theorem.

For the second statement of the theorem differentiate a second time and rearrange.
EXERCISE 5
The probability density function of the beta distribution is given by
\[ f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad \alpha > 0, \beta > 0, \quad 0 < x < 1. \]
where,
\[ B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \]
Show that \( EX^n = \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}. \)

EXERCISE 6
Show that
1. \( \Gamma(\alpha + 1) = \alpha \Gamma(\alpha). \)
2. \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \)

EXERCISE 7
Let \( X \sim N(\mu, \sigma). \)

a. Use the properties of moment generating functions to show that \( aX + b \sim N(a\mu + b, a\sigma). \)

b. Use the cdf method to show that \( aX + b \sim N(a\mu + b, a\sigma). \)

EXERCISE 8
Answer the following questions:

a. Let \( \ln(X) \sim N(\mu, \sigma). \) Find \( EX \) and \( var(X). \)

b. Let \( X_1, X_2, \ldots, X_n \) be independent random variables having respectively the normal distributions \( N(\mu_i, \sigma_i), i = 1, \ldots, n. \) Consider the random variable \( Y = \sum_{i=1}^n k_i X_i. \) Use moment generating functions to find the distribution of \( Y. \)

c. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with \( X_i \sim \Gamma(\alpha, \beta). \) Use the properties of moment generating functions to find the distribution of \( T = X_1 + X_2 + \ldots X_n \) and \( \bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n}. \)

EXERCISE 9
Let \( X \sim N(\mu, \sigma). \) Stein’s lemma states that if \( g \) is a differentiable function satisfying \( Eg'(X) < \infty \) then \( E[g(X)](X - \mu) = \sigma^2 Eg'(X). \) Use Stein’s lemma to show that \( EX^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4. \) Hint: Write \( EX^4 \) as \( EX^3(X - \mu + \mu). \)

EXERCISE 10
Let \( X \) follow a normal distribution with mean \( \mu \) and variance \( \sigma^2. \) Show that the normal pdf is a member of the exponential family. Note: Use \( h(x) = 1. \).