

## HOMEWORK 1 SOLUTIONS

### EXERCISE 1:

$$X \sim \Gamma(\alpha, \theta) \rightarrow f(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \theta^\alpha} \quad \begin{array}{l} x > 0 \\ \alpha > 0 \\ \theta > 0 \end{array}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha) \theta^\alpha} x^\alpha e^{-x/\theta}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha) \theta^\alpha} e^{\ln x^\alpha - x/\theta}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha) \theta^\alpha} e^{\alpha \ln x - \frac{1}{\theta} x}$$

$$h(x) = \frac{1}{x}, \quad c(\theta) = \frac{1}{\Gamma(\alpha) \theta^\alpha}$$

$$w_1(\theta) = \alpha, \quad t_1(x) = \ln x$$

$$w_2(\theta) = \frac{1}{\theta}, \quad t_2(x) = -x$$

EXERCISE 2 :

$$F_Y(y) = P(Y \leq y) = P(CX \leq y) = P\left(X \leq \frac{y}{c}\right)$$

$$F_Y(y) = F_X\left(\frac{y}{c}\right)$$

$$\text{THEN } f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$$

$$f_Y(y) = \frac{1}{c} \frac{\left(\frac{y}{c}\right)^{\alpha-1} e^{-y/c\theta}}{\Gamma(\alpha) \theta^\alpha}$$

$$f_Y(y) = \frac{y^{\alpha-1} e^{-y/c\theta}}{\Gamma(\alpha) (c\theta)^\alpha} \quad \therefore Y \sim T(\alpha, c\theta)$$

EXERCISE 3 :

$$E X^k = \int_0^{\infty} \frac{x^{\alpha+k-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}} dx = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha+k-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+k)\beta^{\alpha+k}}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} \frac{x^{\alpha+k-1} e^{-x/\beta}}{\Gamma(\alpha+k)\beta^{\alpha+k}} dx = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$$

$$k=1 \rightarrow EX = \frac{\Gamma(\alpha+1)\beta}{\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha)\beta}{\Gamma(\alpha)} = \alpha\beta$$

$$k=2 \rightarrow EX^2 = \frac{\Gamma(\alpha+2)\beta^2}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)\Gamma(\alpha)\beta^2}{\Gamma(\alpha)}$$

$$EX^2 = \alpha(\alpha+1)\beta^2$$

$$\text{VAR}(X) = EX^2 - (EX)^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$\text{VAR}(X) = \alpha\beta^2.$$

### EXERCISE 4

$$X \sim b(n, p) \quad P(X) = \binom{n}{x} p^x (1-p)^{n-x}$$

IN EXPONENTIAL FAMILY FORM:

$$P(x) = \binom{n}{x} (1-p)^n e^{x \ln \frac{p}{1-p}}$$

$$h(x) = \binom{n}{x}, \quad c(\theta) = (1-p)^n, \quad w_1(\theta) = \ln \frac{p}{1-p}, \quad t_1(x) = x$$

FROM STATEMENT (a) OF THE THEOREM:

$$E \left[ \frac{1}{P(1-p)} X \right] = - \left( - \frac{n}{1-p} \right) \rightarrow E X = np$$

FROM STATEMENT (b) OF THE THEOREM:

$$\frac{\partial^2 \ln (1-p)^n}{\partial p^2} = - \frac{n}{(1-p)^2}$$

$$\frac{\partial^2 \ln \frac{p}{1-p}}{\partial p^2} = \frac{-1 + 2p}{p^2 (1-p)^2}$$

$$\text{VAR} \left( \frac{1}{P(1-p)} X \right) = \frac{n}{(1-p)^2} + \frac{(1-2p) np}{p^2 (1-p)^2}$$

$$\text{VAR}(X) = np(1-p)$$