University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 1

EXERCISE 1

A pdf or pmf is called an exponential family if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right).$$

Express $X \sim \Gamma(\alpha, \beta)$ in this form.

EXERCISE 2

Suppose $X \sim \Gamma(\alpha, \beta)$. The pdf of $X \sim \Gamma(\alpha, \beta)$ is given by $f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, x > 0, \alpha > 0, \beta > 0$. We want to find the pdf of $Y = cX$, where $c > 0$. Find the cdf of Y and then differentiate it w.r.t. y to find the pdf. Note: The cumulative distribution function (cdf) is defined as $F_Y(y) = P(Y \leq y)$. Replace Y with cX and solve for X. This will give you $F_Y(y) = F_X(\text{some expression of y})$ and then you differentiate. The answer should be that cX also follows a gamma distribution.

EXERCISE 3

Let $X \sim \Gamma(\alpha, \beta)$. Show that $EX^k = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$ and use it to find the mean and variance of X.
Hint 1: The pdf of $X \sim \Gamma(\alpha, \beta)$ is given by $f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, x > 0, \alpha > 0, \beta > 0$. Therefore, $\int_0^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}dx = 1$. We want to find EX^k, which is the expectation of a function of X. Therefore, using $E[g(X)] = \int_x g(x)f(x)dx$ we get $EX^k = \int_0^\infty \frac{x^{\alpha+k-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}dx$. Now you need to make the integral equal to 1 by moving constants outside and multiplying and dividing by other constants.
Hint 2: For EX us $k = 1$. Also, you can use the following property of the gamma function: $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. For the variance, use $k = 2$ to find EX^2 and then $\text{var}(X) = EX^2-(EX)^2$.

EXERCISE 4

The two theorems we discussed in class are:

$$E\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x)\right) = -\frac{\partial}{\partial \theta_j} logc(\boldsymbol{\theta}).$$

and

$$var\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x)\right) = -\frac{\partial^2}{\partial \theta_j^2} logc(\boldsymbol{\theta}) - E\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(x)\right).$$

Note: Here log is the natural logarithm.

Use the theorem for exponential families to find the mean and variance of $X \sim b(n, p)$. Please see the handout on exponential families about the binomial distribution.