EXERCISE 1
Let \( X \sim \Gamma(\alpha, \beta) \). Show that \( EX^k = \frac{\Gamma(\alpha+k)}{\Gamma(k)} \beta^k \) and use it to find the mean and variance of \( X \).

Hint 1: The pdf of \( X \sim \Gamma(\alpha, \beta) \) is given by \( f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} \), \( x > 0, \alpha > 0, \beta > 0 \). Therefore, \( \int_0^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx = 1 \). We want to find \( EX^k \), which is the expectation of a function of \( X \). Therefore, using \( E[g(X)] = \int g(x)f(x)dx \) we get \( EX^k = \int_0^\infty \frac{x^{\alpha+k-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx \). Now you need to make the integral equal to 1 by moving constants outside and multiplying and dividing by other constants.

Hint 2: For \( EX \) use \( k = 1 \). Also, you can use the following property of the gamma function: \( \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \). For the variance, use \( k = 2 \) to find \( EX^2 \) and then \( \text{var}(X) = EX^2 - (EX)^2 \).

EXERCISE 2
Answer the following questions:

a. Use the properties of moment generating functions to show that \( aX + b \sim N(a\mu + b, a\sigma) \).

b. Let \( \ln(X) \sim N(\mu, \sigma) \). Find \( EX \) and \( \text{var}(X) \).

c. Let \( X_1, X_2, \ldots, X_n \) be independent random variables having respectively the normal distributions \( N(\mu_i, \sigma_i), i = 1, \ldots, n \). Consider the random variable \( Y = \sum_{i=1}^n k_i X_i \). Use moment generating functions to find the distribution of \( Y \).

d. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with \( X_i \sim \Gamma(\alpha, \beta) \). Use the properties of moment generating functions to find the distribution of \( T = X_1 + X_2 + \ldots + X_n \) and \( \bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n} \).

EXERCISE 3
The probability density function of the beta distribution is given by \( f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} \), \( \alpha > 0, \beta > 0, 0 < x < 1 \), where, \( B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \). Show that \( EX^n = \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)} \).

EXERCISE 4
Suppose \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with \( X_i \sim \exp(\lambda) \). Show that \( \sum_{i=1}^n X_i \) follows a gamma distribution. What are the parameters? Then use the result of exercise 1 to find \( E \left[ \frac{1}{\sum_{i=1}^n X_i} \right] \).

EXERCISE 5
Answer the following questions:

a. Let \( M_X(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t} \) be the moment-generating function of a discrete random variable \( X \). Find \( EX \) and \( \text{var}(X) \).

b. Suppose \( U \sim \Gamma(\alpha, \beta) \), with \( \alpha > 0, \beta > 0 \) and let \( Y = e^U \). Suppose we want to find \( EY \) and \( \text{var}(Y) \). One way to do this is to find first the pdf of \( Y \) and then compute the moments using \( EY = \int_y yf(y)dy \) and \( EY^2 = \int_y y^2f(y)dy \). Instead, use properties of moment generating function to find without integration \( EY \) and \( \text{var}(Y) \).

c. Let \( X \) follow the Poisson probability distribution with parameter \( \lambda \). Its moment-generating function is \( M_X(t) = e^{\lambda(e^t-1)} \). Show that the moment-generating function of \( Z = \frac{X-\lambda}{\sqrt{\lambda}} \) is given by \( M_Z(t) = e^{-\sqrt{\lambda}t}e^{\lambda(e^{\sqrt{\lambda}t}-1)} \). Then use the series expansion of \( e^{\sqrt{\lambda}t} = 1 + \frac{\sqrt{\lambda}t}{1!} + \left(\frac{\sqrt{\lambda}t}{2!}\right)^2 + \left(\frac{\sqrt{\lambda}t}{3!}\right)^3 + \cdots \) to show that \( \lim_{\lambda \to \infty} M_Z(t) = e^{\frac{t^2}{2}} \). In other words, as \( \lambda \to \infty \), the ratio \( Z = \frac{X-\lambda}{\sqrt{\lambda}} \) converges to the standard normal distribution.