EXERCISE 1
Let $X \sim N(\mu, \sigma)$.

a. Use the properties of moment generating functions to show that $aX + b \sim N(a\mu + b, a\sigma)$.

b. Use the cdf method to show that $aX + b \sim N(a\mu + b, a\sigma)$.

EXERCISE 2

Answer the following questions:

a. Let $\ln(X) \sim N(\mu, \sigma)$. Find $E(X)$ and $\text{var}(X)$.

b. Let $X_1, X_2, \ldots, X_n$ be independent random variables having respectively the normal distributions $N(\mu_i, \sigma_i), i = 1, \ldots, n$. Consider the random variable $Y = \sum_{i=1}^{n} k_i X_i$. Use moment generating functions to find the distribution of $Y$.

c. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$. Use the properties of moment generating functions to find the distribution of $T = X_1 + X_2 + \ldots X_n$ and $\bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n}$.

EXERCISE 3

Let $X \sim N(\mu, \sigma)$. Stein’s lemma states that if $g$ is a differentiable function satisfying $Eg'(X) < \infty$ then $E[g(X)(X - \mu)] = \sigma^2 Eg'(X)$. Use Stein’s lemma to show that $EX^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$. Hint: Write $EX^4$ as $EX^3(X - \mu + \mu)$.

EXERCISE 4

Let $X_1, \ldots, X_n$ i.i.d. random variables with $X_i \sim N(\mu, \sigma)$. Express the vector $\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$ in the form $AX$ and find its mean and variance covariance matrix. Show some typical elements of the variance covariance matrix.

EXERCISE 5

Answer the following questions:

a. Suppose $X$ has a uniform distribution on $(0, 1)$. Find the mean and variance covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$.

b. Suppose $X_1$ and $X_2$ are independent with $\Gamma(\alpha, 1)$ and $\Gamma(\alpha + \frac{1}{2}, 1)$ distributions. Let $Y = 2\sqrt{X_1X_2}$. Find $E(Y)$ and $\text{var}(Y)$.

EXERCISE 6

Answer the following questions:

a. Let $X = (X_1, \ldots, X_n)'$ be a random vector with joint moment generating function $M_X(t)$. In class we discuss this theorem: Let $M_i(t) = \frac{\partial M_X(t)}{\partial t_i}, M_{ii}(t) = \frac{\partial^2 M_X(t)}{\partial t_i^2}$, and $M_{ij}(t) = \frac{\partial^2 M_X(t)}{\partial t_i \partial t_j}$. Then, $EX_i = M_i(0), EX_i^2 = M_{ii}(0)$, and $EX_iX_j = M_{ij}(0)$. Prove this theorem when $n = 2$.

b. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Find the probability density function of $Y$. Find $E(Y)$ and $\text{var}(Y)$.