University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 2

EXERCISE 1

Let $X \sim N(\mu, \sigma)$.

a. Use the properties of moment generating functions to show that $aX + b \sim N(a\mu + b, a\sigma)$.

b. Use the cdf method to show that $aX + b \sim N(a\mu + b, a\sigma)$.

EXERCISE 2

Answer the following questions:

a. Let $\ln(X) \sim N(\mu, \sigma)$. Find EX and $var(X)$.

b. Let $X_1, X_2, \ldots, X_n$ be independent random variables having respectively the normal distributions $N(\mu_i, \sigma_i), i = 1, \ldots, n$. Consider the random variable $Y = \sum_{i=1}^{n} k_i X_i$. Use moment generating functions to find the distribution of Y.

c. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$. Use the properties of moment generating functions to find the distribution of $T = X_1 + X_2 + \ldots X_n$ and $\bar{X} = \frac{X_1 + X_2 + \ldots X_n}{n}$.

EXERCISE 3

Let $X \sim N(\mu, \sigma)$. Stein's lemma states that if g is a differentiable function satisfying $Eg'(X) < \infty$ then $E\left[g(X)(X - \mu)\right] = \sigma^2 Eg'(X)$. Use Stein's lemma to show that $EX^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$. Hint: Write EX^4 as $EX^3(X - \mu + \mu)$.

EXERCISE 4

Let $X_1, \ldots, X_n$ i.i.d. random variables with $X_i \sim N(\mu, \sigma)$. Express the vector $\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$ in the form $\mathbf{AX}$ and find its mean and variance covariance matrix. Show some typical elements of the variance covariance matrix.

EXERCISE 5

Answer the following questions:

a. Suppose X has a uniform distribution on $(0, 1)$. Find the mean and variance covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$.

b. Suppose X_1 and X_2 are independent with $\Gamma(\alpha, 1)$ and $\Gamma(\alpha + \frac{1}{2}, 1)$ distributions. Let $Y = 2\sqrt{X_1 X_2}$. Find EY and $var(Y)$.

EXERCISE 6

Answer the following questions.

a. Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a random vector with joint moment generating function $M_{\mathbf{X}}(\mathbf{t})$. In class we discuss this theorem: Let $M_i(\mathbf{t}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $M_{ii}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial {t_i}^2}$, and $M_{ij}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$. Then, $EX_i = M_i(\mathbf{0})$, $EX_i^2 = M_{ii}(\mathbf{0})$, and $EX_iX_j = M_{ij}(\mathbf{0})$. Prove this theorem when $n = 2$.

b. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Find the probability density function of Y. Find EY and $var(Y)$.

EXERCISE 7

Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y) = \theta y^{\theta-1}, 0 < y < 1, \theta > 0$. Let $W_i = -ln(Y_i)$. Find the pdf of W_i. Show that $2\theta \sum_{i=1}^{n} W_i$ follows a gamma distribution. What are the parameters of this distribution.

EXERCISE 8

Let X follow a normal distribution with mean μ and variance σ^2. Show that the normal pdf is a member of the exponential family. Note: Use $h(x) = 1$.