University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 2

EXERCISE 1
Let $X \sim \Gamma(\alpha, \beta)$. Show that $EX^k = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$ and use it to find the mean and variance of X.

Hint 1: The pdf of $X \sim \Gamma(\alpha, \beta)$ is given by $f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, x > 0, \alpha > 0, \beta > 0$. Therefore, $\int_0^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}dx = 1$. We want to find EX^k, which is the expectation of a function of X. Therefore, using $E[g(X)] = \int_x g(x)f(x)dx$ we get $EX^k = \int_0^\infty \frac{x^{\alpha+k-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}dx$. Now you need to make the integral equal to 1 by moving constants outside and multiplying and dividing by other constants.
Hint 2: For EX use $k = 1$. Also, you can use the following property of the gamma function: $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. For the variance, use $k = 2$ to find EX^2 and then $\text{var}(X) = EX^2 - (EX)^2$.

EXERCISE 2
Answer the following questions:

a. Let $X \sim N(\mu, \sigma)$. Use the properties of moment generating functions to show that if a, b are constants then $aX + b \sim N(a\mu + b, a\sigma)$.

b. Let $\ln(X) \sim N(\mu, \sigma)$. Find EX and $var(X)$.

c. Let $X_1, X_2, \ldots, X_n$ be independent random variables having respectively the normal distributions $N(\mu_i, \sigma_i), i = 1, \ldots, n$. Consider the random variable $Y = \sum_{i=1}^n k_i X_i$. Use moment generating functions to find the distribution of Y.

d. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$. Use the properties of moment generating functions to find the distribution of $T = X_1 + X_2 + \ldots X_n$ and $\bar{X} = \frac{X_1+X_2+\ldots X_n}{n}$.

EXERCISE 3
The probability density function of the beta distribution is given by $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},\ \alpha > 0,\ \beta > 0,\ 0 < x < 1$. where, $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Show that $EX^n = \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$.

EXERCISE 4
Suppose $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim exp(\lambda)$. Show that $\sum_{i=1}^n X_i$ follows a gamma distribution. What are the parameters? Then use the result of exercise 1 to find $E\left[\frac{1}{\sum_{i=1}^n X_i}\right]$.

EXERCISE 5
Answer the following questions:

a. Let $M_X(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$ be the moment-generating function of a discrete random variable X. Find $E(X)$ and $var(X)$.

b. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Suppose we want to find EY and $var(Y)$. One way to do this is to find first the pdf of Y and then compute the moments using $EY = \int_y yf(y)dy$ and $EY^2 = \int_y y^2 f(y)dy$. Instead, use properties of moment generating function to find without integration EY and $var(Y)$.

c. Let X follow the Poisson probability distribution with parameter λ. Its moment-generating function is $M_X(t) = e^{\lambda(e^t-1)}$. Show that the moment-generating function of $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is given by $M_Z(t) = e^{-\sqrt{\lambda}t}e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}$. Then use the series expansion of $e^{\frac{t}{\sqrt{\lambda}}} = 1 + \frac{\frac{t}{\sqrt{\lambda}}}{1!} + \frac{(\frac{t}{\sqrt{\lambda}})^2}{2!} + \frac{(\frac{t}{\sqrt{\lambda}})^3}{3!} + \cdots$ to show that $\lim_{\lambda\to\infty} M_Z(t) = e^{\frac{1}{2}t^2}$. In other words, as $\lambda \to \infty$, the ratio $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ converges to the standard normal distribution.

EXERCISE 6
Suppose $(Y_1, Y_2, \ldots, Y_n)'$ is a random vector with mean $\mu\mathbf{1}$ and variance covariance matrix $\sigma^2\mathbf{V}$, where $\mathbf{V}$ is an $n \times n$ symmetric matrix of known constants. Consider the expressions
(a) $m = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}$ and
(b) $q = \frac{(\mathbf{Y}-m\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-m\mathbf{1})}{n}$.
Find the mean and variance of (a) and the mean of (b).

EXERCISE 7
Answer the following questions:

a. Suppose Y follows a $\Gamma(\alpha, \beta)$ distribution. Find the mean and variance covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$, where $X = e^Y$.

b. Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with $X_i \sim exp(\lambda)$. Find the expected value and variance of $\frac{1}{\bar{X}}$, where $\bar{X}$ is the sample mean of $X_1, X_2, \ldots, X_n$.

EXERCISE 8
Suppose $Y_1, \ldots, Y_n$ are i.i.d. random variables with $Y_i \sim N(\mu, \sigma)$. Express the following vector in the form $\mathbf{AY}$ and find its mean and variance: $\begin{pmatrix} \bar{Y} \\ Y_1 - Y_2 \\ Y_2 - Y_3 \\ \vdots \\ Y_{n-1} - Y_n \end{pmatrix}$.

EXERCISE 9
Answer the following questions:

a. Let $X \sim \Gamma(\frac{n}{2}, \beta)$. Find the distribution of $Y = \frac{2X}{\beta}$ using the method of cdf and the method of moment generating functions.

b. Suppose X has the p.d.f. $f(x) = 4x^3, 0 < x < 1$. Use the method of cdf to show that $Y = -2\ln X^4$ follows a gamma distribution. What are the parameters of this gamma distribution?

EXERCISE 10
Suppose that $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are two samples, with $X_i \sim N(\mu_1, \sigma_1)$ and $Y_i \sim N(\mu_2, \sigma_2)$. The difference between the sample means, $\bar{X} - \bar{Y}$, is then a linear combination of $m + n$ normal random variables. All the random variables are independent. Answer the following questions:

a. Use moment generating functions to show that $\bar{X} - \bar{Y}$ follows a normal distribution. Find the mean and variance of this distribution.

b. Suppose $\sigma_1^2 = 2, \sigma_2^2 = 2.5$, and $m = n$. Find the sample size n so that $\bar{X} - \bar{Y}$ will be within one unit of $\mu_1 - \mu_2$ with probability 0.95. You can use the standard normal table from the course website here:

`http://www.stat.ucla.edu/~nchristo/statistics100B/stat100b_z_table.pdf.`