University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 2

EXERCISE 1

Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim N(\mu, \sigma)$.

a. Express the vector $\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$ in the form $\mathbf{AX}$ and find its mean and variance covariance matrix. Hint: Express $\bar{X}$ in the form $\mathbf{a}'\mathbf{X}$. What is the vector $\mathbf{a}$ here? Show some typical elements of the variance covariance matrix.

b. Use properties of the trace to find $E[\mathbf{X}'\mathbf{AX}]$. Hint: From class notes we see that when $\mathbf{X}'\mathbf{AX}$ is a scalar we can write it as $tr(\mathbf{X}'\mathbf{AX})$ and then take expectation.

EXERCISE 2

Answer the following questions:

a. Suppose X has a uniform distribution on $(0, 1)$. Find the mean and variance covariance matrix of the random vector $\begin{pmatrix} X \\ X^2 \end{pmatrix}$. Hint: Review concepts of $E(X)$ and $E(X^2)$ for the mean vector. For the variance covariance matrix we will need to find also $cov(X, X^2)$.

b. Suppose X_1 and X_2 are independent with $\Gamma(\alpha, 1)$ and $\Gamma(\alpha + \frac{1}{2}, 1)$ distributions. Let $Y = 2\sqrt{X_1 X_2}$. Find EY and $var(Y)$. Hint: Here X_1 and X_2 are independent. Also the result from homework 1, exercise 3 can be very helpful.

EXERCISE 3

Answer the following questions.

a. Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a random vector with joint moment generating function $M_{\mathbf{X}}(\mathbf{t})$. In class we discuss this theorem: Let $M_i(\mathbf{t}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $M_{ii}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial {t_i}^2}$, and $M_{ij}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$. Then, $EX_i = M_i(\mathbf{0})$, $EX_i^2 = M_{ii}(\mathbf{0})$, and $EX_iX_j = M_{ij}(\mathbf{0})$. Prove this theorem when $n = 2$. Hint: Use the definition of the joint moment generating function (double integral when $n = 2$) and take first derivative w.r.t. t_1, second derivative w.r.t. t_1, and second derivative w.r.t. t_1 and t_2 and evaluate these derivatives by setting $t_1 = 0$ and $t_2 = 0$.

b. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Find the probability density function of Y. Find EY and $var(Y)$. Hint: Use the moment generating function of the gamma distribution.

EXERCISE 4

Suppose the joint moment generating function of X and Y is given by $M_{X,Y}(t_1, t_2) = e^{8t_1 + 3t_2 + \frac{1}{2}5t_1^2 + 2t_1t_2 + 2t_2^2}$. Use the Corollary from the handout on joint moment generating functions to find the following:

a. Mean vector of $\begin{pmatrix} X \\ Y \end{pmatrix}$.

b. Variance covariance matrix of $\begin{pmatrix} X \\ Y \end{pmatrix}$.

c. $\mathrm{corr}(X, Y)$

d. Are X, Y independent? (Find the marginal moment generating functions of X and Y and check if their product is equal to the joint moment generating function.).

EXERCISE 5

Answer the following questions:

a. Suppose $X \sim \Gamma(\frac{1}{2}, 2)$. Find the pdf of $Y = X^{\frac{1}{4}}$.

b. Suppose $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim exp(\lambda)$. Find the distribution of $\sum_{i=1}^{n} X_i$.

c. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y) = \theta y^{\theta-1}, 0 < y < 1, \theta > 0$. Let $W_i = -ln(Y_i)$. Find the pdf of W_i.

d. Refer to question (d). Show that $2\theta \sum_{i=1}^{n} W_i$ follows a gamma distribution. What are the parameters of this distribution.

e. Refer to question (d). Find $E\left(\frac{n-1}{\sum_{i=1}^{n} W_i}\right)$, where $W_i = -ln(Y_i)$.