University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 3

EXERCISE 1
Suppose Y_1 is the length of time a machine operates without failure. Let Y_2 be the length of time required to repair the machine after a failure. If Y_1 and Y_2 are independent exponential random variables with pdf $f(y_i) = e^{-y_i}, i = 1, 2$, show that the probability density function of $U = \frac{Y_1}{Y_1+Y_2}$ follows the uniform distribution. Hint: Let $V = Y_2$ and find the joint of U and V. Finally integrate the joint w.r.t. V to find the pdf of U. Also, show that U and $Y_1 + Y_2$ are independent.

EXERCISE 2
Let X and Y be independent standard normal random variables. Consider the transformation $U = X + Y$ and $V = X - Y$.

a. Use the theory of distributions of functions of random variables (Jacobian) to find the joint pdf of U and V. Are U and V independent? Why? What is the distribution of U and the distribution of V?

b. Use moment generating functions to answer question (a).

EXERCISE 3
Let $X \sim \text{beta}(\alpha, \beta)$ and Let $Y \sim \text{beta}(\alpha+\beta, \gamma)$. If X and Y are independent random variables find the joint pdf of U and V where $U = XY$ and $V = X$.

EXERCISE 4
Suppose Y is the number of pollution particles in volume v and assume that Y follows the Poisson distribution with mean λv.

a. If a point is randomly selected within the volume v find the pdf of its distance R from the nearest pollution particle. Note the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

b. Refer to question (a). Let $U = R^3$. Show that U follows exponential distribution. What is the mean and variance of U?

EXERCISE 5
Let X_1, X_2, and X_3 be three independent gamma random variables with parameters $(\frac{r_1}{2}, 2), (\frac{r_2}{2}, 2), (\frac{r_3}{2}, 2)$ respectively. Use the joint distribution of functions of random variables and the Jacobian to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent and that $Y_2 \sim \Gamma(\frac{r_1+r_2}{2}, 2)$. Conclude by reasoning that $\frac{\frac{X_1}{r_1}}{\frac{X_2}{r_2}}$ and $\frac{\frac{X_3}{r_3}}{\frac{X_1+X_2}{r_1+r_2}}$ are independent.