University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 3

EXERCISE 1

Suppose Y_1 is the length of time a machine operates without failure. Let Y_2 be the length of time required to repair the machine after a failure. If Y_1 and Y_2 are independent exponential random variables with pdf $f(y_i) = e^{-y_i}, i = 1, 2$, show that the probability density function of $U = \frac{Y_1}{Y_1+Y_2}$ follows the uniform distribution. Hint: Let $V = Y_2$ and find the joint of U and V. Finally integrate the joint w.r.t. V to find the pdf of U. Also, show that U and $Y_1 + Y_2$ are independent.

EXERCISE 2

Let X and Y be independent standard normal random variables. Consider the transformation $U = X + Y$ and $V = X - Y$.

a. Use the theory of distributions of functions of random variables (Jacobian) to find the joint pdf of U and V. Are U and V independent? Why? What is the distribution of U and the distribution of V?

b. Use moment generating functions to answer question (a).

EXERCISE 3

Let $X \sim \text{beta}(\alpha, \beta)$ and Let $Y \sim \text{beta}(\alpha+\beta, \gamma)$. If X and Y are independent random variables find the joint pdf of U and V where $U = XY$ and $V = X$.

EXERCISE 4

Suppose Y is the number of pollution particles in volume v and assume that Y follows the Poisson distribution with mean λv.

a. If a point is randomly selected within the volume v find the pdf of its distance R from the nearest pollution particle. Note the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

b. Refer to question (a). Let $U = R^3$. Show that U follows exponential distribution. What is the mean and variance of U?

EXERCISE 5

Let X_1, X_2, and X_3 be three independent gamma random variables with parameters $(\frac{r_1}{2}, 2), (\frac{r_2}{2}, 2), (\frac{r_3}{2}, 2)$ respectively. Use the joint distribution of functions of random variables and the Jacobian to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent and that $Y_2 \sim \Gamma(\frac{r_1+r_2}{2}, 2)$. Conclude by reasoning that $\frac{\frac{X_1}{r_1}}{\frac{X_2}{r_2}}$ and $\frac{\frac{X_3}{r_3}}{\frac{X_1+X_2}{r_1+r_2}}$ are independent.

EXERCISE 6

The multinomial distribution is defined as follows: A sequence of n independent experiments is performed and each experiment can result in one of r possible outcomes with probabilities $p_1, p_2, \ldots, p_r$ with $\sum_{i=1}^{r} p_i = 1$. Let X_i be the number of the n experiments that result in outcome i, $i = 1, 2, \ldots, r$. Then, $P(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{x_1!x_2!\cdots x_r!}p_1^{x_1}p_2^{x_2}\cdots p_r^{x_r}$. Find the conditional pmf of $(X_1, X_2, \ldots, X_{r-1})$ given that $X_r = x_r$ and show that it is also multinomial. What are the parameters of this pmf?

EXERCISE 7

Answer the following questions:

a. Consider independent Bernoulli trials where the success probability follows a beta distribution: $P_i \sim beta(\alpha, \beta)$. Therefore, $X_i|P_i \sim \text{Bernoulli}(P_i)$. Suppose we are interested in the total number of successes $Y = \sum_{i=1}^{n} X_i$. Find the mean and variance of Y by using the following theorem on expectation and variance by conditioning:
Theorem: If X and Y are any two random variables then

1. $E[X] = E(E[X|Y])$.
2. $\text{var}[X] = E(\text{var}[X|Y]) + \text{var}(E[X|Y])$.

b. Refer to question (a). Suppose now we have k independent binomial experiments with different fixed number of trials $(n_1, \ldots, n_k)$ and success probability that follows a beta distribution with parameters α, β. So we can write, $X_i|P_i \sim \text{b}(n_i, P_i)$ and $P_i \sim \text{beta}(\alpha, \beta)$. Use the theorem on conditional mean and variance from question (b) to find the mean and variance of Y, where $Y = \sum_{i=1}^{k} X_i$.

EXERCISE 8

Find the pdf of $\Pi_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are independent uniform (0,1).