University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 4

Answer the following questions:

a. Let $X \sim F_{m,n}$. Show that $F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$ and $t^2_{1-\frac{\alpha}{2};n} = F_{1-\alpha;1,n}$ and draw the relevant graphs for both questions.

b. Let $X_1, X_2, \ldots, X_{13}$ and $Y_1, Y_2, \ldots, Y_{16}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = \frac{1}{5}\sigma_2^2$, but σ_2^2 is unknown. Construct a ratio that follows the t distribution with 27 degrees of freedom.

c. Find the mean and variance of $X \sim F_{n,m}$.

d. Let $\mathbf{X} \sim N_n(\mu \mathbf{1}, \mathbf{\Sigma})$, where $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, and $\mathbf{\Sigma}$ is the variance covariance matrix of $\mathbf{X}$.

Let $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, with $\rho > -\frac{1}{n-1}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Therefore, when $\rho = 0$ we have $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{I})$, and in this case we showed in class that $\bar{X}$ and $\sum_{i=1}^n (X_i - \bar{X})^2$ are independent. Are they independent when $\rho \neq 0$?

e. If $\mathbf{Y} \sim N_2(\mathbf{0}, \mathbf{\Sigma})$ prove that $\left(\mathbf{Y}'\mathbf{\Sigma^{-1}}\mathbf{Y} - \frac{Y_1^2}{\sigma_1^2}\right) \sim \chi_1^2$.

Note: $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$, $\mathbf{\Sigma^{-1}} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}$, $\sigma_{12} = \sigma_{21}$, $\sigma_{12} = \rho\sigma_1\sigma_2$.

f. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be a random sample from a bivariate normal distribution, $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. (Note: $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent). What is the distribution of $n\left(\bar{X} - \mu_1, \bar{Y} - \mu_2\right)\mathbf{\Sigma^{-1}}\begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix}$.