University of California, Los Angeles
Department of Statistics

Statistics 100B Instructor: Nicolas Christou

Homework 4

EXERCISE 1
Let \((X_1, Y_1), \ldots, (X_n, Y_n)\), be a random sample from a bivariate normal distribution with parameters \(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\). (Note: \((X_1, Y_1), \ldots, (X_n, Y_n)\) are independent). What is the joint distribution of \((\bar{X}, \bar{Y})\)? Hint: Find the joint moment generating function of \((\bar{X}, \bar{Y})\) and compare it to the joint moment generating function of multivariate normal distribution.

EXERCISE 2
Answer the following questions:

a. Let \(X\) and \(Y\) follow the bivariate normal distribution with parameters \(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\). Show that \(W = X - \mu_1\) and \(Q = (Y - \mu_2) - \rho \sigma_2 \sigma_1 (X - \mu_1)\) are independent normal random variables.

b. Let \(X_1\) and \(X_2\) be two independent normal random variables with mean zero and variance 1. Show that the vector \(Z = (Z_1, Z_2)'\), where

\[
Z_1 = \mu_1 + \sigma_1 X_1 \\
Z_2 = \mu_2 + \rho \sigma_2 X_1 + \sigma_2 \sqrt{1 - \rho^2} X_2
\]

follows the bivariate normal distribution with parameters \(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\).

c. Let \(X_1, X_2, X_3\) be i.i.d. random variables \(N(0,1)\). Show that \(Y_1 = X_1 + \delta X_3\) and \(Y_2 = X_2 + \delta X_3\) have bivariate normal distribution. Find the value of \(\delta\) so that the correlation coefficient between \(Y_1\) and \(Y_2\) is \(\rho = \frac{1}{2}\).

EXERCISE 3
Let \(X \sim N_n(\mu_1, \Sigma)\), where \(1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\), and \(\Sigma\) is the variance covariance matrix of \(X\). Let \(\Sigma = (1 - \rho) I + \rho J\), with \(\rho > -\frac{1}{n-1}\), \(I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}\) and \(J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}\). Therefore, when \(\rho = 0\) we have \(X \sim N_n(\mu_1, I)\), and in this case we showed in class that \(\bar{X}\) and \(\sum_{i=1}^n (X_i - \bar{X})^2\) are independent. Are they independent when \(\rho \neq 0\)?

EXERCISE 4
Suppose \(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2(\mu, \Sigma)\). Consider the vector \(\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\), where \(Y_i = e^{X_i}, i = 1, 2\). Find \(EY_1^3\) and covariance between \(Y_1^3, Y_2^3\).
EXERCISE 5
Let $X_1, X_2, X_3$ be i.i.d. $N(0, 1)$ random variables. Suppose $Y_1 = X_1 + X_2 + X_3, Y_2 = X_1 - X_2, Y_3 = X_1 - X_3$. Find the joint pdf of $Y = (Y_1, Y_2, Y_3)'$ using:

a. The method of variable transformations (Jacobian).

b. Multivariate normal distribution properties.

EXERCISE 6
Let $(X_i, Y_i), i = 1, 2, \ldots, n$, be a random sample from a bivariate normal distribution. Note: $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent. Find the distribution of the vector

$$
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n \\
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{pmatrix}.
$$

EXERCISE 7
Let $Y \sim N_n(\mu, \Sigma)$, where $Y$, $\mu$, and $\Sigma$ are partitioned as follows

$$
Y = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},
$$

and let $A$ be a matrix such that $A\Sigma_{22} = \Sigma_{12}$.

a. Let $U = Q_1 - AQ_2$ and $V = Q_2$. Find the joint distribution of $U$ and $V$.

b. Show that $Q_1 | Q_2 \sim N(\mu_1 + A(Q_2 - \mu_2), \Sigma_{11} - A\Sigma_{22}A')$.

EXERCISE 8
Suppose the following random variables for adult men

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>Percent body fat from Siri’s (1956) equation</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>Age (years)</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>Weight (lbs)</td>
</tr>
<tr>
<td>$Y_4$</td>
<td>Height (inches)</td>
</tr>
<tr>
<td>$Y_5$</td>
<td>Abdomen circumference (cm)</td>
</tr>
</tbody>
</table>

follow multivariate normal with mean vector and variance covariance matrix as follows:

Mean vector:

<table>
<thead>
<tr>
<th></th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
<th>y4</th>
<th>y5</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>y1</td>
<td>70</td>
<td>31</td>
<td>151</td>
<td>-3</td>
</tr>
<tr>
<td>179</td>
<td>y2</td>
<td>31</td>
<td>154</td>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>70</td>
<td></td>
<td>33</td>
<td>283</td>
<td></td>
<td></td>
</tr>
<tr>
<td>93</td>
<td>y4</td>
<td>-3</td>
<td>-8</td>
<td>33</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>y5</td>
<td>74</td>
<td>31</td>
<td>283</td>
<td>3</td>
</tr>
</tbody>
</table>

and variance-covariance matrix:

<table>
<thead>
<tr>
<th>y2</th>
<th>31</th>
<th>154</th>
<th>-2</th>
<th>-8</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>y3</td>
<td>151</td>
<td>-2</td>
<td>866</td>
<td>33</td>
<td>283</td>
</tr>
<tr>
<td>y4</td>
<td>-3</td>
<td>-8</td>
<td>33</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>y5</td>
<td>74</td>
<td>31</td>
<td>283</td>
<td>3</td>
<td>117</td>
</tr>
</tbody>
</table>

Find the conditional distribution of the percent body fat $Y_1$ given $Y_2, Y_3, Y_4$. 