HOMEWORK 4 SOLUTIONS

EXERCISE 1:

$$\underset{\sim}{W} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad E\underset{\sim}{W} = \begin{pmatrix} \mu_1 \underset{\sim}{1} \\ \mu_2 \underset{\sim}{1} \end{pmatrix}$$

$$\operatorname{var}(\underset{\sim}{W}) = \begin{pmatrix} \sigma_1^2 \underset{\sim}{I} & \sigma_{12} \underset{\sim}{I} \\ \sigma_{12} \underset{\sim}{I} & \sigma_2^2 \underset{\sim}{I} \end{pmatrix}$$

EXERCISE 2:

$Y_i = \varepsilon_i + c\,\varepsilon_{i-1}$

$$\left.\begin{array}{l} Y_1 = \varepsilon_1 + c\,\varepsilon_0 \\ Y_2 = \varepsilon_2 + c\,\varepsilon_1 \\ Y_3 = \varepsilon_3 + c\,\varepsilon_2 \\ \quad\vdots \qquad \vdots \qquad \vdots \\ Y_n = \varepsilon_n + c\,\varepsilon_{n-1} \end{array}\right\} \rightarrow \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 \\ 0 & c & 1 & 0 & \cdots & 0 \\ 0 & 0 & c & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$\underset{\sim}{Y} = \underset{\sim}{A}\,\underset{\sim}{\varepsilon}$ WHERE $\underset{\sim}{\varepsilon} \sim N_{n+1}(\underset{\sim}{0}, \sigma^2 \underset{\sim}{I})$

$\therefore \underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, \sigma^2 \underset{\sim}{A}\underset{\sim}{A}')$

Exercise 3

$$X_1 = Y_1 Y_3$$

$$X_2 = Y_2 Y_3$$

$$X_3 = Y_3 - Y_1Y_3 - Y_2Y_3 = Y_3(1 - Y_1 - Y_2)$$

JOINT PDF OF X_1, X_2, X_3 IS:

$$f(x_1, x_2, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3)$$

$$= \frac{x_1^{\alpha_1 - 1} e^{-x_1}}{\Gamma(\alpha_1)} \cdot \frac{x_2^{\alpha_2 - 1} e^{-x_2}}{\Gamma(\alpha_2)} \cdot \frac{x_3^{\alpha_3 - 1} e^{-x_3}}{\Gamma(\alpha_3)}$$

JACOBIAN:

$$J = \begin{vmatrix} \frac{\partial Y_1}{\partial X_1} & \frac{\partial Y_1}{\partial X_2} & \frac{\partial Y_1}{\partial X_3} \\ \frac{\partial Y_2}{\partial X_1} & \frac{\partial Y_2}{\partial X_2} & \frac{\partial Y_2}{\partial X_3} \\ \frac{\partial Y_3}{\partial X_1} & \frac{\partial Y_3}{\partial X_2} & \frac{\partial Y_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} \frac{X_2+X_3}{(X_1+X_2+X_3)^2} & -\frac{X_1}{(X_1+X_2+X_3)^2} & -\frac{X_1}{(X_1+X_2+X_3)^2} \\ \frac{\cancel{X_1+X_2}}{(X_1+X_2+X_3)^2} & \frac{\cancel{X_1}+X_3}{(X_1+X_2+X_3)^2} & -\frac{X_2}{(X_1+X_2+X_3)^2} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{y_3^2}.$$

WE CAN ALSO COMPUTE THE JACOBIAN USING

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \\ \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \end{vmatrix} = \begin{vmatrix} Y_3 & 0 & Y_1 \\ 0 & Y_3 & Y_2 \\ -Y_3 & -Y_3 & 1-Y_1-Y_2 \end{vmatrix} = y_3^2$$

$\rightarrow$

$$f(y_1,y_2,y_3) = \frac{(y_1y_3)^{\alpha_1-1} e^{-y_1y_3} (y_2y_3)^{\alpha_2-1} e^{-y_2y_3} \left(y_3(1-y_1-y_2)\right)^{\alpha_3-1} e^{-y_3(1-y_1-y_2)}}{\Gamma(\alpha_1)\quad\Gamma(\alpha_2)\quad\Gamma(\alpha_3)} y_3^2$$

$$= \frac{y_1^{\alpha_1-1} y_2^{\alpha_2-1} (1-y_1-y_2)^{\alpha_3-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \cdot y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3} = *$$

<u>NOTE</u>: $\displaystyle\int_0^\infty \frac{y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} dy_3 = 1$

$$\hookrightarrow y_3^{\alpha_1+\alpha_2+\alpha_3-1} e^{-y_3} = \Gamma(\alpha_1+\alpha_2+\alpha_3)$$

$$* = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1-1} y_2^{\alpha_2-1} (1-y_1-y_2)^{\alpha_3-1}$$

Exercise 4

$$M_{Q_1,Q_2}(t,s) = E e^{tQ_1 + sQ_2}$$

$$= E e^{t\bar{X} + s(X_1 - \bar{X})} = E e^{t\frac{X_1 + \dots + X_n}{n} + s\left(X_1 - \frac{X_1 + \dots + X_n}{n}\right)}$$

$$= E e^{X_1\left(\frac{t}{n} + s - \frac{s}{n}\right) + X_2\left(\frac{t}{n} - \frac{s}{n}\right) + \dots + X_n\left(\frac{t}{n} - \frac{s}{n}\right)}$$

$$= M_{X_1}\left(\frac{t}{n} + s - \frac{s}{n}\right) \cdot \left\{M_{X_i}\left(\frac{t}{n} - \frac{s}{n}\right)\right\}^{n-1}$$

WHERE $M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

FOR X_1 REPLACE t WITH $\frac{t}{n} + s - \frac{s}{n}$

FOR THE OTHER $n-1$ X_i's REPLACE t WITH $\frac{t}{n} - \frac{s}{n}$

SIMILARLY,

$$M_{\bar{X}}(t) = E e^{t\bar{X}} = \left(M_{X_i}\left(\frac{t}{n}\right)\right)^n$$

$$M_{X_1 - \bar{X}}(s) = E e^{s(X_1 - \bar{X})} = E e^{X_1\left(s - \frac{s}{n}\right) + X_2\left(-\frac{s}{n}\right) + \dots + X_n\left(-\frac{s}{n}\right)}$$

$$= M_{X_1}\left(s - \frac{s}{n}\right) \cdot \left(M_{X_i}\left(-\frac{s}{n}\right)\right)^{n-1}$$

Exercise 5

$$\bar{Y} = \frac{1}{n}\mathbf{1}'Y$$

$$\begin{bmatrix} Y_1-\bar{Y} \\ Y_2-\bar{Y} \\ \vdots \\ Y_n-\bar{Y} \end{bmatrix} = \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)Y$$

$$\begin{bmatrix} \frac{1}{n}\mathbf{1}' \\ I - \frac{1}{n}\mathbf{1}\mathbf{1}' \end{bmatrix} Y = AY$$

$$Y \sim N_n\left[\mu\mathbf{1}, \sigma^2\left((1-\rho)I + \rho\mathbf{1}\mathbf{1}'\right)\right]$$

$$\operatorname{var}(AY) = A \operatorname{var}(Y) A'$$

$$= \begin{bmatrix} \frac{1}{n}\cdot\mathbf{1}' \\ I - \frac{1}{n}\mathbf{1}\mathbf{1}' \end{bmatrix} \sigma^2\left[(1-\rho)I + \rho\mathbf{1}\mathbf{1}'\right]\left(\frac{1}{n}\mathbf{1}, \; I - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)$$

CHECK COVARIANCE:

$$\frac{1-\rho}{n}\mathbf{1}' + \frac{\rho}{n}\mathbf{1}'\mathbf{1}\mathbf{1}' - (1-\rho)\frac{1}{n^2}\mathbf{1}'\mathbf{1}\mathbf{1}' - \rho\frac{1}{n^2}\mathbf{1}'\mathbf{1}\mathbf{1}'\mathbf{1}\mathbf{1}'$$

$$= \mathbf{0}'$$

Yes, Ybar and sum(Yi-Ybar)^2 are independent

because Ybar is independent of the vector of the deviations.