University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 4

EXERCISE 1

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be a random sample from a bivariate normal distribution with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$. (Note: $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent). What is the joint distribution of $(\bar{X}, \bar{Y})$? Hint: Find the joint moment generating function of $(\bar{X}, \bar{Y})$ and compare it to the joint moment generating function of multivariate normal distribution.

EXERCISE 2

Answer the following questions:

a. Let X and Y follow the bivariate normal distribution with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$. Show that $W = X - \mu_1$ and $Q = (Y - \mu_2) - \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1)$ are independent normal random variables.

a. Let X_1 and X_2 be two independent normal random variables with mean zero and variance 1. Show that the vector $\mathbf{Z} = (Z_1, Z_2)'$, where

$$\begin{aligned} Z_1 &= \mu_1 + \sigma_1 X_1 \\ Z_2 &= \mu_2 + \rho\sigma_2 X_1 + \sigma_2\sqrt{1-\rho^2}X_2 \end{aligned}$$

follows the bivariate normal distribution with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$.

EXERCISE 3

Let $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{\Sigma})$, where $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, and $\mathbf{\Sigma}$ is the variance covariance matrix of $\mathbf{X}$. Let $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, with $\rho > -\frac{1}{n-1}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Therefore, when $\rho = 0$ we have $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{I})$, and in this case we showed in class that $\bar{X}$ and $\sum_{i=1}^n (X_i - \bar{X})^2$ are independent. Are they independent when $\rho \neq 0$?

EXERCISE 4

Suppose X_1, X_2, X_3 be independent random variables that follow $\Gamma(\alpha_i, 1), i = 1, 2, 3$ distribution. Let

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2 + X_3} \\ Y_2 &= \frac{X_2}{X_1 + X_2 + X_3} \\ Y_3 &= X_1 + X_2 + X_3 \end{aligned}$$

denote 3 new random variables. Show that the joint pdf of Y_1, Y_2, Y_3 is given by

$$f(y_1, y_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1}.$$

(Random variables that have a joint pdf of this form follow the Dirichlet distribution.)