University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 4

EXERCISE 1

Let $(X_i, Y_i), i = 1, 2, \ldots, n$ be a random sample from a bivariate normal distribution (the n pairs are independent). Consider the vector $\mathbf{W} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$, where $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$. Find the distribution of $\mathbf{W}$.

EXERCISE 2

Suppose $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma)$ and let $Y_i = \epsilon_i + c\epsilon_{i-1}$, where c is a known constant. Show that $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ follows multivariate normal. What is the mean and variance covariance matrix?

EXERCISE 3

Suppose X_1, X_2, X_3 be independent random variables that follow $\Gamma(\alpha_i, 1), i = 1, 2, 3$ distribution. Let

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2 + X_3} \\ Y_2 &= \frac{X_2}{X_1 + X_2 + X_3} \\ Y_3 &= X_1 + X_2 + X_3 \end{aligned}$$

denote 3 new random variables. Show that the joint pdf of Y_1, Y_2 is given by

$$f(y_1, y_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1}.$$

(Random variables that have a joint pdf of this form follow the Dirichlet distribution.)

EXERCISE 4

Let $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma)$. Let $Q_1 = \bar{X}$ and $Q_2 = X_1 - \bar{X}$. Show that $M_{Q_1, Q_2(t,s)} = M_{Q_1}(t) M_{Q_2}(s)$.

EXERCISE 5

Let $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{\Sigma})$, where $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, and $\mathbf{\Sigma}$ is the variance covariance matrix of $\mathbf{X}$. Let $\mathbf{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, with $\rho > -\frac{1}{n-1}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Therefore, when $\rho = 0$ we have $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{I})$, and in this case we showed in class that $\bar{X}$ and $\sum_{i=1}^n (X_i - \bar{X})^2$ are independent. Are they independent when $\rho \neq 0$?