University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 5

Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ be i.i.d.Poisson(λ) and let $\bar{X}$ and S^2 be the sample mean and sample variance respectively. Each one of these two estimators has expected value equal to λ (why?). Which estimator is better?

b. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both $\bar{X}$ and cS are unbiased estimators of θ, where $c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$. Show that for any α the estimator $\alpha\bar{X}+(1-\alpha)cS$ is also unbiased estimator of θ. For what value of α this estimator has the minimum variance?

c. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with α known. Find an unbiased estimator of $\frac{1}{\beta}$. (Find $E\frac{1}{\bar{X}}$ and then adjust it to be unbiased of $\frac{1}{\beta}$.)

d. A general technique for reducing bias in an estimator is the following. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, and let $\hat{\theta}$ be some estimator of a parameter θ. In order to reduce the bias the method works as follows: We calculate $\hat{\theta}^{(i)}$, $i = 1, 2, \ldots, n$ just as $\hat{\theta}$ is calculated but using the $n-1$ observations with X_i removed from the sample. This new estimator is given by $\hat{\theta}^* = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^n \hat{\theta}^{(i)}$. To apply this concept we will use the Bernoulli distribution. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p). It is given that the MLE of p^2 is $\hat{\theta} = \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2$. Show that $\hat{\theta}$ is not unbiased for p^2.

e. Refer to question (d). Use the technique described above to reduce the bias in $\hat{\theta}$. Does the method remove the bias entirely in this example?

f. Find the Rao-Cramér lower bound of an estimator of θ but do not assume that $\hat{\theta}$ is unbiased estimator of θ. Please show the entire derivation.

g. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with α known. Is $\hat{\beta} = \frac{\bar{X}}{\alpha}$ efficient estimator of β?