Answer the following questions:

a. Let $X \sim F_{m,n}$. Show that $F_{\alpha; m,n} = \frac{1}{F_{1-\alpha; n,m}}$ and $t_{1-\frac{\alpha}{2}; m}^2 = F_{1-\alpha; 1,n}$ and draw the relevant graphs for both questions.

b. Let $X_1, X_2, \ldots, X_{13}$ and $Y_1, Y_2, \ldots, Y_{16}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = \frac{1}{5} \sigma_2^2$, but $\sigma_2^2$ is unknown. Construct a ratio that follows the $t$ distribution with 27 degrees of freedom.

c. Find the mean and variance of $X \sim F_{n,m}$.

d. If $Y \sim N_2(0, \Sigma)$ prove that $\left( Y' \Sigma^{-1} Y - \frac{Y^2}{\sigma_1^2} \right) \sim \chi_1^2$.

Note: $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$, $\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_2^2} \sigma_{12}^2 - \sigma_{12}^2 \\ \sigma_{21}^2 - \sigma_{12}^2 \end{pmatrix}$, $\sigma_{12} = \sigma_{21}$, $\sigma_1 = \rho \sigma_1 \sigma_2$.

e. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be a random sample from a bivariate normal distribution, $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. (Note: $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent). What is the distribution of $n \left( \bar{X} - \mu_1, \bar{Y} - \mu_2 \right) \Sigma^{-1} \begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix}$.

f. Let $X \sim F_{k,m}$. Find $EX^r$. 