Statistics 100B  
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Homework 5

Answer the following questions:

a. Let $X \sim F_{m,n}$. Show that $F_{\alpha; m,n} = \frac{1}{F_{1 - \alpha; m,n}}$ and $t_{1 - \frac{\alpha}{2}; n}^2 = F_{1 - \alpha; 1,n}$ and draw the relevant graphs for both questions.

b. Let $X_1, X_2, \ldots, X_{13}$ and $Y_1, Y_2, \ldots, Y_{16}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = \frac{1}{5} \sigma_2^2$, but $\sigma_2^2$ is unknown. Construct a ratio that follows the $t$ distribution with 27 degrees of freedom.

c. Find the mean and variance of $X \sim F_{n,m}$.

d. Let $X \sim N_n(\mu I, \Sigma)$, where $I = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, and $\Sigma$ is the variance covariance matrix of $X$.

Let $\Sigma = (1-\rho)I + \rho J$, with $\rho > -\frac{1}{n-1}$, $I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Therefore, when $\rho = 0$ we have $X \sim N_n(\mu I, I)$, and in this case we showed in class that $\overline{X}$ and $\sum_{i=1}^n (X_i - \overline{X})^2$ are independent. Are they independent when $\rho \neq 0$?

e. If $Y \sim N_2(0, \Sigma)$ prove that $(Y' \Sigma^{-1} Y - \frac{Y^2}{\sigma^2}) \sim \chi^2_1$.

Note: $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$, $\Sigma^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}$, $\sigma_{12} = \sigma_{21}$, $\sigma_{12} = \rho \sigma_1 \sigma_2$.

f. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be a random sample from a bivariate normal distribution, $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. (Note: $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent). What is the distribution of $n \left( \overline{X} - \mu_1, \overline{Y} - \mu_2 \right) \Sigma^{-1} \begin{pmatrix} \overline{X} - \mu_1 \\ \overline{Y} - \mu_2 \end{pmatrix}$.