University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 5

Answer the following questions:

a. Let $X \sim F_{m,n}$. Show that $F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$ and $t^2_{1-\frac{\alpha}{2};n} = F_{1-\alpha;1,n}$ and draw the relevant graphs for both questions.

b. Let $X_1, X_2, \ldots, X_{13}$ and $Y_1, Y_2, \ldots, Y_{16}$ represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = \frac{1}{5}\sigma_2^2$, but σ_2^2 is unknown. Construct a ratio that follows the t distribution with 27 degrees of freedom.

c. Derive the distribution of the sample mean $\bar{X}$ of independent $X_1, \ldots, X_n$ where, $X_i \sim \Gamma(\alpha, \beta)$. Find a transformation of $\bar{X}$ that follows a χ^2 distribution. What are the degrees of freedom of this transformation?

d. If $\mathbf{Y} \sim N_2(\mathbf{0}, \mathbf{\Sigma})$ prove that $\left(\mathbf{Y}'\mathbf{\Sigma^{-1}}\mathbf{Y} - \frac{Y_1^2}{\sigma_1^2}\right) \sim \chi^2_1$.
Note: $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$, $\mathbf{\Sigma^{-1}} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}$, $\sigma_{12} = \sigma_{21}$, $\sigma_{12} = \rho\sigma_1\sigma_2$.

e. Find the mean and variance of the $X \sim t_n$ distribution.

f. Find the mean and variance of the $X \sim F_{n,m}$ distribution.

g. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$, be a random sample from a bivariate normal distribution, with $E(X_i) = \mu_1, E(Y_i) = \mu_2, \text{var}(X_i) = \sigma_1^2, \text{var}(Y_i) = \sigma_2^2$, and $\text{corr}(X_i, Y_i) = \rho$.
Note: $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent.
What is the distribution of $n\left(\bar{X} - \mu_1, \bar{Y} - \mu_2\right)\mathbf{\Sigma^{-1}}\begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix}$.

h. Let $Z \sim N(0,1)$ and $U\chi^2_n$. Assume that Z and U are independent. Let $x = \frac{z}{\sqrt{\frac{U}{n}}}$ and $w = u$. Find the joint pdf of X and W and then find the marginal pdf of X. Your answer should be the same as the pdf given on page 7 of handout #13.

i. Find the mean and variance of the non-central t distribution with non-centrality parameter δ.

j. Find the mean and variance of the non-central F distribution with non-centrality parameter θ and degrees of freedom n_1 for the numerator and n_2 for the denominator.