

University of California, Los Angeles  
Department of Statistics

Statistics 100B

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Homework 5

Answer the following questions:

- a. Let  $X \sim F_{m,n}$ . Show that  $F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$  and  $t_{1-\frac{\alpha}{2};n}^2 = F_{1-\alpha;1,n}$  and draw the relevant graphs for both questions.
- b. Let  $X_1, X_2, \dots, X_{13}$  and  $Y_1, Y_2, \dots, Y_{16}$  represent two independent random samples from the respective normal distributions  $N(\mu_1, \sigma_1)$  and  $N(\mu_2, \sigma_2)$ . It is given that  $\sigma_1^2 = \frac{1}{5}\sigma_2^2$ , but  $\sigma_2^2$  is unknown. Construct a ratio that follows the  $t$  distribution with 27 degrees of freedom.
- c. Derive the distribution of the sample mean  $\bar{X}$  of independent  $X_1, \dots, X_n$  where,  $X_i \sim \Gamma(\alpha, \beta)$ . Find a transformation of  $\bar{X}$  that follows a  $\chi^2$  distribution. What are the degrees of freedom of this transformation?
- d. If  $\mathbf{Y} \sim N_2(\mathbf{0}, \Sigma)$  prove that  $\left(\mathbf{Y}'\Sigma^{-1}\mathbf{Y} - \frac{Y_1^2}{\sigma_1^2}\right) \sim \chi_1^2$ .  
Note:  $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$ ,  $\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}$ ,  $\sigma_{12} = \sigma_{21}$ ,  $\sigma_{12} = \rho\sigma_1\sigma_2$ .
- e. Find the mean and variance of the  $X \sim t_n$  distribution.
- f. Find the mean and variance of the  $X \sim F_{n,m}$  distribution.
- g. Let  $(X_1, Y_1), \dots, (X_n, Y_n)$ , be a random sample from a bivariate normal distribution, with  $E(X_i) = \mu_1$ ,  $E(Y_i) = \mu_2$ ,  $\text{var}(X_i) = \sigma_1^2$ ,  $\text{var}(Y_i) = \sigma_2^2$ , and  $\text{corr}(X_i, Y_i) = \rho$ .  
Note:  $(X_1, Y_1), \dots, (X_n, Y_n)$  are independent.  
What is the distribution of  $n \begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix} \Sigma^{-1} \begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix}$ .
- h. Let  $Z \sim N(0, 1)$  and  $U \sim \chi_n^2$ . Assume that  $Z$  and  $U$  are independent. Let  $x = \frac{z}{\sqrt{\frac{U}{n}}}$  and  $w = u$ . Find the joint pdf of  $X$  and  $W$  and then find the marginal pdf of  $X$ . Your answer should be the same as the pdf given on page 7 of handout #13.
- i. Find the mean and variance of the non-central  $t$  distribution with non-centrality parameter  $\delta$ .
- j. Find the mean and variance of the non-central  $F$  distribution with non-centrality parameter  $\theta$  and degrees of freedom  $n_1$  for the numerator and  $n_2$  for the denominator.