Statistics 100B

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Homework 6

Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson($\lambda$) and let $\bar{X}$ and $S^2$ be the sample mean and sample variance respectively. Each one of these two estimators has expected value equal to $\lambda$ (why?). Which estimator is better?

b. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both $\bar{X}$ and $cS$ are unbiased estimators of $\theta$, where $c = \frac{\sqrt{n-1} \Gamma \left(\frac{n-1}{2}\right)}{\sqrt{2^n \Gamma \left(\frac{n}{2}\right)}}$. Show that for any $\alpha$ the estimator $\alpha \bar{X} + (1-\alpha)cS$ is also unbiased estimator of $\theta$. For what value of $\alpha$ this estimator has the minimum variance?

c. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with $\alpha$ known. Find an unbiased estimator of $\frac{1}{\beta}$. (Find $E \frac{1}{\bar{X}}$ and then adjust it to be unbiased of $\frac{1}{\beta}$.)

d. Let $X_1, X_2, X_3, X_4$ be i.i.d. random variables with $X_i \sim N(0, 1)$. Show that

$$\sum_{i=1}^{4} (X_i - \bar{X})^2 = \frac{(X_1 - X_2)^2}{2} + \frac{(X_3 - \frac{(X_1 + X_2)}{2})^2}{\frac{3}{2}} + \frac{(X_4 - \frac{(X_1 + X_2 + X_3)}{3})^2}{\frac{4}{3}}.$$

Show that the three terms in the RHS are independent each with a $\chi^2_1$ distribution.