University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 6

EXERCISE 1
Answer the following questions:

a. Let X_1, X_2, and X_3 be three independent gamma random variables with parameters $(\frac{r_1}{2}, 2), (\frac{r_2}{2}, 2), (\frac{r_3}{2}, 2)$ respectively. Use the joint distribution of functions of random variables and the Jacobian to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent and that $Y_2 \sim \Gamma(\frac{r_1+r_2}{2}, 2)$. Conclude by reasoning that $\frac{\frac{X_1}{r_1}}{\frac{X_2}{r_2}}$ and $\frac{\frac{X_3}{r_3}}{\frac{X_1+X_2}{r_1+r_2}}$ are independent.

b. Let $\mathbf{X} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $M_{X_1,X_2}(t_1, t_2)$ be the joint moment generating function of X_1 and X_2. Compute the covariance between X_1 and X_2 by using $\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M(0,0)}{\partial t_1} \frac{\partial M(0,0)}{\partial t_2}$. Now let $\psi(t_1, t_2) = lnM_{X_1,X_2}(t_1, t_2)$. Show that the covariance can be computed directly using $\frac{\partial^2 \psi(0,0)}{\partial t_1 \partial t_2}$.

EXERCISE 2
Answer the following questions:

a. Let $X_1, \ldots, X_n$ have a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and variance covariance matrix $\boldsymbol{\Sigma}$. Let $Y = \mathbf{c}'\mathbf{X}$ and $Z = \mathbf{d}'\mathbf{X}$ be two linear combinations of $X_1, \ldots, X_n$, where $\mathbf{c}' = (c_1, \ldots, c_n)$ and $\mathbf{d}' = (d_1, \ldots, d_n)$ are constants. Find the joint moment generating function $M_{Y,Z}(t_1, t_2)$ to see that Y and Z follow bivariate normal distribution.

b. Prove that Y and Z are independent if $\mathbf{c}'\boldsymbol{\Sigma}\mathbf{d} = 0$. If $X_1, \ldots, X_n$ are independent normal random variables with same variance σ^2 what is the necessary and sufficient condition for independence between Y and Z?

EXERCISE 3
Let X_1, X_2, X_3 be i.i.d. $N(0, 1)$ random variables. Suppose $Y_1 = X_1 + X_2 + X_3, Y_2 = X_1 - X_2, Y_3 = X_1 - X_3$. Find the joint pdf of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ using:

a. The method of variable transformations (Jacobian).

b. Multivariate normal distribution properties.