EXERCISE 1
Answer the following questions:

a. Let $X_1, X_2,$ and $X_3$ be three independent gamma random variables with parameters $(\frac{r_1}{2}, 2), (\frac{r_2}{2}, 2), (\frac{r_3}{2}, 2)$ respectively. Use the joint distribution of functions of random variables and the Jacobian to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent and that $Y_2 \sim \Gamma(\frac{r_1+r_2}{2}, 2)$. Conclude by reasoning that $\frac{X_1}{X_2}$ and $\frac{X_3}{X_1+X_2}$ are independent.

b. Let $X \sim N_2(\mu, \Sigma)$ and let $M_{X_1, X_2}(t_1, t_2)$ be the joint moment generating function of $X_1$ and $X_2$. Compute the covariance between $X_1$ and $X_2$ by using $\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M(0,0)}{\partial t_1} \frac{\partial M(0,0)}{\partial t_2}$. Now let $\psi(t_1, t_2) = ln M_{X_1, X_2}(t_1, t_2)$. Show that the covariance can be computed directly using $\frac{\partial^2 \psi(0,0)}{\partial t_1 \partial t_2}$.

EXERCISE 2
Answer the following questions:

a. Let $X_1, \ldots, X_n$ have a multivariate normal distribution with mean vector $\mu$ and variance covariance matrix $\Sigma$. Let $Y = c'X$ and $Z = d'X$ be two linear combinations of $X_1, \ldots, X_n$, where $c' = (c_1, \ldots, c_n)$ and $d' = (d_1, \ldots, d_n)$ are constants. Find the joint moment generating function $M_{Y,Z}(t_1, t_2)$ to see that $Y$ and $Z$ follow bivariate normal distribution.

b. Prove that $Y$ and $Z$ are independent if $c' \Sigma d = 0$. If $X_1, \ldots, X_n$ are independent normal random variables with same variance $\sigma^2$ what is the necessary and sufficient condition for independence between $Y$ and $Z$?

EXERCISE 3
Let $X_1, X_2, X_3$ be i.i.d. $N(0, 1)$ random variables. Suppose $Y_1 = X_1 + X_2 + X_3, Y_2 = X_1 - X_2, Y_3 = X_1 - X_3$. Find the joint pdf of $Y = (Y_1, Y_2, Y_3)'$ using:

a. The method of variable transformations (Jacobian).

b. Multivariate normal distribution properties.