Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson($\lambda$) and let $\bar{X}$ and $S^2$ be the sample mean and sample variance respectively. Each one of these two estimators has expected value equal to $\lambda$ (why?). Which estimator is better?

b. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta^2)$, $\theta > 0$. For this model both $\bar{X}$ and $cS$ are unbiased estimators of $\theta$, where $c = \frac{\sqrt{n-1} \Gamma\left(\frac{n}{2}\right)}{\sqrt{2} \Gamma\left(\frac{n+1}{2}\right)}$. Show that for any $\alpha$ the estimator $\alpha \bar{X} + (1-\alpha)cS$ is also unbiased estimator of $\theta$. For what value of $\alpha$ this estimator has the minimum variance?

c. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with $\alpha$ known. Find an unbiased estimator of $\frac{1}{\beta}$. (Find $E\frac{1}{\bar{X}}$ and then adjust it to be unbiased of $\frac{1}{\beta}$.)

d. A general technique for reducing bias in an estimator is the following. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, and let $\hat{\theta}$ be some estimator of a parameter $\theta$. In order to reduce the bias the method works as follows: We calculate $\hat{\theta}^{(i)}$, $i = 1, 2, \ldots, n$ just as $\hat{\theta}$ is calculated but using the $n-1$ observations with $X_i$ removed from the sample. This new estimator is given by $\hat{\theta}^* = n\hat{\theta} - \frac{n-1}{n} \sum_{i=1}^{n} \hat{\theta}^{(i)}$. To apply this concept we will use the Bernoulli distribution. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli($p$). It is given that the MLE of $p^2$ is $\hat{\theta} = \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2$. Show that $\hat{\theta}$ is not unbiased for $p^2$.

e. Refer to question (d). Use the technique described above to reduce the bias in $\hat{\theta}$. Does the method remove the bias entirely in this example?

f. Find the Rao-Cramér lower bound of an estimator of $\theta$ but do not assume that $\hat{\theta}$ is unbiased estimator of $\theta$. Please show the entire derivation.

g. Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \Gamma(\alpha, \beta)$ with $\alpha$ known. Is $\hat{\beta} = \frac{\bar{X}}{\alpha}$ efficient estimator of $\beta$?