

University of California, Los Angeles
Department of Statistics

Statistics 100B

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Homework 7

EXERCISE 1

Let Y_1, \dots, Y_n be i.i.d. random variables with pdf $f(y|\theta) = (\frac{2y}{\theta})\exp(-\frac{y^2}{\theta}), y > 0$ (Weibull distribution). Show that $\hat{\theta} = \frac{\sum_{i=1}^n Y_i^2}{n}$ is unbiased estimator of θ . Is $\hat{\theta}$ an efficient estimator of θ ?

EXERCISE 2

Let X_1, \dots, X_n be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both \bar{X} and cS are unbiased estimators of θ , where $c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$. Define the estimator $T = \alpha_1\bar{X} + \alpha_2(cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T - \theta)^2$.

EXERCISE 3

Answer the following questions:

- a. If X_1, \dots, X_n are independent random variables with $X_i \sim N(\mu_i, \sigma)$ and if $r = [(X_1 - \mu_1)^2 + \dots + (X_n - \mu_n)^2]^{\frac{1}{2}}$, show that $E(r) = \sqrt{2}\sigma \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}n)}$.
- b. Consider the following two independent sets of random variables: Let X_1, \dots, X_n i.i.d. random variables with $X_i \sim N(\mu_1, \sigma)$ and let Y_1, \dots, Y_m i.i.d. random variables with $Y_i \sim N(\mu_2, \sigma)$. Consider the random variable $S_p^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^m (Y_i - \bar{Y})^2}{n+m-2}$. Find ES_p .

EXERCISE 4

Answer the following questions:

- a. Let X_1, X_2, \dots, X_n denote a random sample from a normal population with mean zero and unknown variance σ^2 . Consider the estimator $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2$. Is it efficient?
- b. Consider the following two independent samples X_1, \dots, X_m and Y_1, \dots, Y_n selected from two exponential distributions with parameters λ_1 and λ_2 . Find c such that the distribution of $c\frac{\bar{X}}{\bar{Y}}$ is F . What are the degrees of freedom?

EXERCISE 5

Suppose Y_1, Y_2, \dots, Y_n follow multivariate normal with mean $\mu\mathbf{1}$ and variance covariance matrix $\sigma^2\mathbf{V}$, where \mathbf{V} is an $n \times n$ symmetric matrix of known constants. The maximum likelihood estimates of μ and σ^2 are $\hat{\mu} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}$ and $\hat{\sigma}^2 = \frac{(\mathbf{Y} - \hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y} - \hat{\mu}\mathbf{1})}{n}$. Find the information matrix $\mathbf{I}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\mu, \sigma^2)'$. Is $\hat{\mu}$ an efficient estimator of μ ?

EXERCISE 6

Let X_1, X_2, \dots, X_n denote an i.i.d. random sample from the following distribution ($\alpha > 0$).

$$f(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{3^\alpha}, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of X .

Derive the method of moments estimator of α .

Derive the method of maximum likelihood estimate of α .

EXERCISE 7

Let X_1, \dots, X_n be i.i.d. $U(0, \alpha)$ and let $X_{(1)}, \dots, X_{(n)}$ be the order statistics. Define the range as $R = X_{(n)} - X_{(1)}$ and the midrange as $Q = \frac{X_{(1)} + X_{(n)}}{2}$. Find the joint pdf of R, Q .