University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 7

EXERCISE 1

Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y|\theta) = (\frac{2y}{\theta})exp(-\frac{y^2}{\theta}), y > 0$ (Weibull distribution). Show that $\hat{\theta} = \frac{\sum_{i=1}^n Y_i^2}{n}$ is unbiased estimator of θ. Is $\hat{\theta}$ an efficient estimator of θ?

EXERCISE 2

Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both $\bar{X}$ and cS are unbiased estimators of θ, where $c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$. Define the estimator $T = \alpha_1 \bar{X} + \alpha_2 (cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T-\theta)^2$.

EXERCISE 3

Answer the following questions:

a. If $X_1, \ldots, X_n$ are independent random variables with $X_i \sim N(\mu_i, \sigma)$ and if $r = [(X_1-\mu_1)^2 + \ldots + (X_n - \mu_n)^2]^{\frac{1}{2}}$, show that $E(r) = \sqrt{2}\sigma\frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n)}$.

b. Consider the following two independent sets of random variables: Let $X_1, \ldots, X_n$ i.i.d. random variables with $X_i \sim N(\mu_1, \sigma)$ and let Let $Y_1, \ldots, Y_m$ i.i.d. random variables with $Y_i \sim N(\mu_2, \sigma)$. Consider the random variable $S_p^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^m (Y_i - \bar{Y})^2}{n+m-2}$. Find ES_p.

EXERCISE 4

Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal population with mean zero and unknown variance σ^2. Consider the estimator $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$. Is it efficient?

b. Consider the following two independent samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ selected from two exponential distributions with parameters λ_1 and λ_2. Find c such that the distribution of $c\frac{\bar{X}}{\bar{Y}}$ is F. What are the degrees of freedom?

EXERCISE 5

Suppose $Y_1, Y_2, \ldots, Y_n$ follow multivariate normal with mean $\mu\mathbf{1}$ and variance covariance matrix $\sigma^2\mathbf{V}$, where $\mathbf{V}$ is an $n \times n$ symmetric matrix of known constants. Show that the maximum likelihood estimates of μ and σ^2 are $\hat{\mu} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}$ and $\hat{\sigma}^2 = \frac{(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})}{n}$. Find the information matrix $\mathbf{I}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\mu, \sigma^2)'$. Is $\hat{\mu}$ an efficient estimator of μ?

EXERCISE 6

Let $X_1, X_2, \cdots, X_n$ denote an i.i.d. random sample from the following distribution ($\alpha > 0$).

$$f(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{3^\alpha}, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of X.
Derive the method of moments estimator of α.
Derive the method of maximum likelihood estimate of α.

EXERCISE 7

Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \alpha)$ and let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics. Define the range as $R = X_{(n)} - X_{(1)}$ and the midrange as $Q = \frac{X_{(1)} + X_{(n)}}{2}$. Find the joint pdf of R, Q.

EXERCISE 8

Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim N(\mu, \sigma)$. Find c so that $E[cS^2 - \sigma^2]^2$ is minimized. How is this new estimator cS^2 compared to S^2?