EXERCISE 1
Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y|\theta) = \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}}$, $y > 0$ (Weibull distribution).
Show that $\hat{\theta} = \frac{\sum_{i=1}^{n} Y_i^2}{n}$ is unbiased estimator of $\theta$. Is $\hat{\theta}$ an efficient estimator of $\theta$?

EXERCISE 2
Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta)$, $\theta > 0$. For this model both $\bar{X}$ and $cS$ are unbiased estimators of $\theta$, where $c = \sqrt{\frac{n-1}{\sqrt{2\Gamma\left(\frac{n+1}{2}\right)}}}$. Define the estimator $T = \alpha_1 \bar{X} + \alpha_2 (cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T - \theta)^2$.

EXERCISE 3
Suppose $Y_1, Y_2, \ldots, Y_n$ follow multivariate normal with mean $\mu_1$ and variance covariance matrix $\sigma^2 V$, where $V$ is an $n \times n$ symmetric matrix of known constants. Show that the maximum likelihood estimates of $\mu$ and $\sigma^2$ are $\hat{\mu} = \frac{1}{V^{-1}} \frac{1}{n} Y$ and $\hat{\sigma^2} = \frac{(Y - \hat{\mu})' V^{-1} (Y - \hat{\mu})}{n}$. Find the information matrix $I(\theta)$, where $\theta = (\mu, \sigma^2)'$. Is $\hat{\mu}$ an efficient estimator of $\mu$?

EXERCISE 4
Let $X_1, X_2, \cdots, X_n$ denote an i.i.d. random sample from the following distribution ($\alpha > 0$).
\[
f(x) = \begin{cases} \frac{ax^{a-1}}{3^a}, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}
\]
Find the expected value of $X$.
Derive the method of moments estimator of $a$.
Derive the method of maximum likelihood estimate of $a$.

EXERCISE 5
Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \alpha)$ and let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics. Define the range as $R = X_{(n)} - X_{(1)}$ and the midrange as $Q = \frac{X_{(n)} + X_{(1)}}{2}$. Find the joint pdf of $R, Q$.

EXERCISE 6
Let $X_1, \ldots, X_n$ be i.i.d. random variables with $X_i \sim N(\mu, \sigma)$. Answer the following questions:

a. Verify that $\hat{\theta} = \bar{X} S^2$ is an unbiased estimator of $\theta = \mu \sigma^2$. Find the variance of $\hat{\theta}$.

b. Find an unbiased estimator of $\theta = \frac{\mu}{\sigma^2}$. Find the variance of this estimator.

EXERCISE 7
Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, 1)$ and let $U_1, \ldots, U_n$ be i.i.d. from $U(0,1)$. All the $2n$ random variables are independent. Let $Y_i = X_i U_i, i = 1, \ldots, n$. If the $X_i$ and $U_i$ are both observed, then $\bar{X}$ would be a natural estimator for $\theta$. If only the products $Y_1, \ldots, Y_n$ are observed, then $2\bar{Y}$ may be a reasonable estimator for $\theta$. Are the two estimators unbiased? Determine the relative efficiency of $2\bar{Y}$ with respect to $\bar{X}$. Which estimator is more efficient?
EXERCISE 8
Let $X_1, \ldots, X_n$ be i.i.d. $N(0, \sigma)$. Show that $\hat{\theta} = \frac{C}{n} \sum_{i=1}^{n} \sqrt{X_i^2}$ is a consistent estimator of $\sigma$ if and only if $C = \sqrt{\frac{2}{\pi}}$. Show that the MLE of $\sigma$ is given by $\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}$ and determine the asymptotic relative efficiency of $\hat{\theta}$ with $C = \sqrt{\frac{2}{\pi}}$ compared to $\hat{\sigma}$ the MLE of $\sigma$. Hint: Use the asymptotic efficiency of maximum likelihood estimates.

EXERCISE 9
Let $X_1, \ldots, X_m$ be i.i.d. random variables from an exponential distribution with parameter $\lambda_x$ and $Y_1, \ldots, Y_n$ be i.i.d. random variables from an exponential distribution with parameter $\lambda_y$. The two samples are independent. Answer the following questions.

a. Find an unbiased estimator of the ratio $\frac{\lambda_x}{\lambda_y}$. Your answer should be a function of $\sum_{i=1}^{m} X_i$ and $\sum_{i=1}^{n} Y_i$.

b. Use MSE to find the best estimator of $\frac{\lambda_x}{\lambda_y}$ of the form $\hat{\delta} = c \bar{Y} \bar{X}$.

EXERCISE 10
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables on the interval $[0, 1]$ with pdf

$$f(x) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha - 1} (1 - x)^{2\alpha - 1}, \quad \alpha > 0.$$  

Find the method of moments estimator of $\alpha$.

b. The numbers $w_1, w_2, \ldots, w_n$ are known positive values. The random variables $X_1, X_2, \ldots, X_n$ are independent, and the distribution of $X_i$ is $N(\mu, \sigma \sqrt{w_i})$. Both parameters $\mu$ and $\sigma$ are unknown. Find the maximum likelihood estimates of $\mu$ and $\sigma^2$. Is $\hat{\mu}$ an efficient estimator of $\mu$?

c. An electronic system consists of four components connected in series as shown in the figure below. The components function independently and each one has a lifetime (in weeks) that follows the exponential distribution with $\lambda = \frac{1}{20}$. Find the probability density function of the lifetime of this electronic system and use it to compute the system’s expected lifetime.

![Series System Diagram]

\begin{align*}
1 & \quad A \quad B \quad C \quad D \quad 2
\end{align*}

d. Refer to part (c). If the four components are connected in parallel as shown in the next figure, find the probability density function of the lifetime of this electronic system.

![Parallel System Diagram]