Answer the following questions:

a. Suppose \(Y_1, Y_2, \ldots, Y_n\) follow multivariate normal with mean \(\mu\) and variance covariance matrix \(\sigma^2 V\), where \(V\) is an \(n \times n\) symmetric matrix of known constants. Show that the maximum likelihood estimates of \(\mu\) and \(\sigma^2\) are \(\hat{\mu} = \frac{1}{V^{-1}1}V^{-1}Y\) and \(\hat{\sigma}^2 = \frac{(Y - \hat{\mu}1)'V^{-1}(Y - \hat{\mu}1)}{n}\).

b. Refer to question (a). Find \(E(\hat{\mu})\) and \(E(\hat{\sigma}^2)\).

c. Refer to question (a). Find the Fisher information matrix \(I(\theta)\), where \(\theta = (\mu, \sigma^2)'\). Is \(\hat{\mu}\) an efficient estimator of \(\mu\)?

d. Let \(Y_1, Y_2, \ldots, Y_n\) independent random variables, and let \(Y_i \sim N(i\theta, i\sigma)\), i.e. \(E(Y_i) = i\theta\) and \(\text{var}(Y_i) = i^2\sigma^2\), for \(i = 1, 2, \ldots, n\). Find the maximum likelihood estimator of \(\theta\). Is this estimator efficient estimator of \(\theta\)?

e. Suppose that the radius of a circle is measured with an error \(\epsilon \sim N(0, \sigma)\). If \(n\) independent measurements are made find an unbiased estimator of the area of the circle.

f. Let \(Y_1, \ldots, Y_n\) be i.i.d. random variables from the Weibull distribution \(f(y|\theta) = \left(\frac{2y}{\theta}\right)\exp\left(-\frac{y^2}{\theta}\right), y > 0\). Show that \(\hat{\theta} = \frac{\sum_{i=1}^{n} Y_i^2}{n}\) is unbiased estimator of \(\theta\). Is \(\hat{\theta}\) an efficient estimator of \(\theta\)?

g. Let \(X_1, \ldots, X_n\) be i.i.d. \(N(\theta, \theta), \theta > 0\). For this model both \(\bar{X}\) and \(cS\) are unbiased estimators of \(\theta\), where \(c = \frac{\sqrt{n-1}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2\Gamma\left(\frac{n}{2}\right)}}\). Define the estimator \(T = \alpha_1 \bar{X} + \alpha_2 (cS)\), where we do not assume that \(\alpha_1 + \alpha_2 = 1\). Find the estimator that minimizes \(E(T - \theta)^2\).