Statistics 100B  Instructor: Nicolas Christou

Homework 7

EXERCISE 1
Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with pdf $f(y|\theta) = (\frac{2y}{\theta})exp(-\frac{y^2}{\theta}), y > 0$ (Weibull distribution). Show that $\hat{\theta} = \sum_{i=1}^{n} \frac{Y_i^2}{n}$ is unbiased estimator of $\theta$. Is $\hat{\theta}$ an efficient estimator of $\theta$?

EXERCISE 2
Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta) \theta > 0$. For this model both $\bar{X}$ and $\sqrt{n} \left[\sum_{i=1}^{n} (X_i - \mu)^2\right]^{\frac{1}{2}}$ Where $\alpha = \sqrt{n} \left[\sum_{i=1}^{n} (X_i - \mu)^2\right]^{\frac{1}{2}}$. Define the estimator $T = \alpha_1 \bar{X} + \alpha_2 (cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T - \theta)^2$.

EXERCISE 3
Answer the following questions:

a. If $X_1, \ldots, X_n$ are independent random variables with $X_i \sim N(\mu_i, \sigma)$ and if $r = [(X_1 - \mu_1)^2 + \ldots + (X_n - \mu_n)^2]^{\frac{1}{2}}$, show that $E(r) = \sqrt{2\sigma 1^{\frac{1}{n}+\frac{1}{m}}} 1^{\frac{1}{n+m}}$.

b. Consider the following two independent sets of random variables: Let $X_1, \ldots, X_n$ i.i.d. random variables with $X_i \sim N(\mu_1, \sigma)$ and let $Y_1, \ldots, Y_m$ i.i.d. random variables with $Y_i \sim N(\mu_2, \sigma)$. Consider the random variable $S^2_p = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{m} (Y_i - \bar{Y})^2}{n+m-2}$. Find $ES_p$.

EXERCISE 4
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal population with mean zero and unknown variance $\sigma^2$. Consider the estimator $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2$. Is it efficient?

b. Consider the following two independent samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ selected from two exponential distributions with parameters $\lambda_1$ and $\lambda_2$. Find $c$ such that the distribution of $\frac{X}{Y}$ is F. What are the degrees of freedom?

EXERCISE 5
Let $Y_1, \ldots, Y_n$ be random variables with $E(Y_i) = \mu$, var$(Y_i) = \sigma^2$, and $cov(Y_i, Y_j) = \rho \sigma^2$. The variance covariance matrix is of the form $(a-b)I + bJ$, where $a = 1, b = \rho, J = 11'$. In our model $\Sigma = \sigma^2 [(1-\rho)I + \rho J]$. The inverse of this special matrix can be obtained as follows: $\Sigma^{-1} = \frac{1}{\sigma^2 (1-\rho)} \left[ I - \frac{\rho}{1+(n-1)\rho} J \right]$. Suppose the estimator of $\mu$ is given by $\hat{\mu} = \frac{1}{\Sigma^{-1}1} \Sigma^{-1}Y$. Is $\hat{\mu}$ unbiased estimator of $\mu$? Show that $var(\hat{\mu}) = \frac{1}{\Sigma^{-1}1 \Sigma^{-1}1}$ and simplify it using the inverse of the variance covariance matrix given above. Explain why $\rho > -\frac{1}{n-1}$. 

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EXERCISE 6
Suppose $Y_1, Y_2, \ldots, Y_n$ follow multivariate normal with mean $\mu I$ and variance covariance matrix $\sigma^2 V$, where $V$ is an $n \times n$ symmetric matrix of known constants. The maximum likelihood estimates of $\mu$ and $\sigma^2$ are $\hat{\mu} = \frac{1}{n} V^{-1} Y$ and $\hat{\sigma}^2 = \frac{1}{n} (Y - \hat{\mu}I)' V^{-1} (Y - \hat{\mu}I)$. Find the Fisher information matrix $I(\theta)$, where $\theta = (\mu, \sigma^2)'$. Is $\hat{\mu}$ an efficient estimator of $\mu$?

EXERCISE 7
Let $Y_1, Y_2, \ldots, Y_n$ independent random variables, and let $Y_i \sim N(i\theta, i\sigma)$, i.e. $E(Y_i) = i\theta$ and $\text{var}(Y_i) = i^2 \sigma^2$, for $i = 1, 2, \ldots, n$. Find the maximum likelihood estimator of $\theta$. Is this estimator efficient estimator of $\theta$?

EXERCISE 8
Let $X_1, X_2, \cdots, X_n$ denote an i.i.d. random sample from the following distribution $(\alpha > 0)$.

$$f(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{3^\alpha}, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of $X$.
Derive the method of moments estimator of $\alpha$.
Derive the method of maximum likelihood estimate of $\alpha$. 