Homework 8 solutions

Exercise 1

$\bar{X}, S^2$ ARE INDEPENDENT.

(a). $\hat{\theta} = \bar{X}S^2$

$$E\bar{X}S^2 = (E\bar{X})(ES^2) = \mu\sigma^2$$

$$\text{VAR}(\bar{X}S^2) = E[(\bar{X}S^2)^2] - (E\bar{X}S^2)^2 = *$$

NOTE: $S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$

$$* = [E\bar{X}^2][E(S^2)^2] - (\mu\sigma^2)^2$$

$$= \{\text{VAR}(\bar{X}) + (E\bar{X})^2\} \cdot E(S^2)^2 - \mu^2\sigma^4$$

$$= \left(\frac{\sigma^2}{n} + \mu^2\right) \frac{\Gamma\left(\frac{n-1}{2}+2\right)\left(\frac{2\sigma^2}{n-1}\right)^2}{\Gamma\left(\frac{n-1}{2}\right)} - \mu^2\sigma^4$$

$$= \left(\frac{\sigma^2}{n} + \mu^2\right) \frac{\left(\frac{n-1}{2}+1\right)\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{2\sigma^2}{n-1}\right)^2 - \mu^2\sigma^4$$

$$= \left(\frac{\sigma^2}{n} + \mu^2\right) \frac{(n+1)\sigma^4}{n-1} - \mu^2\sigma^4$$

(b). LET $\hat{\theta} = \dfrac{\bar{X}}{S^2}$ $\quad S^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$

$$E\hat{\theta} = \left(E\bar{X}\right)\left(E(S^2)^{-1}\right) = \mu \, \frac{\Gamma\left(\frac{n-1}{2}-1\right)\left(\frac{2\sigma^2}{n-1}\right)^{-1}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$= \frac{\mu(n-1)}{2\sigma^2} \frac{\Gamma\left(\frac{n-1}{2}-1\right)}{\left(\frac{n-1}{2}-1\right)\Gamma\left(\frac{n-1}{2}-1\right)} = \frac{\mu(n-1)}{2\sigma^2} \cdot \frac{2}{n-3} = \frac{\mu}{\sigma^2}\frac{n-1}{n-3}.$$

THEREFORE, UNBIASED IS $\hat{\theta}^* = \dfrac{n-3}{n-1}\dfrac{\bar{X}}{S^2}$.

$$\text{VAR}\left(\hat{\theta}^*\right) = \left(\frac{n-3}{n-1}\right)^2 \left\{ E\left(\frac{\bar{X}}{S^2}\right)^2 - \left(E\,\frac{\bar{X}}{S^2}\right)^2 \right\}$$

$$= \left(\frac{n-3}{n-1}\right)^2 \left\{ E\bar{X}^2 \cdot E(S^2)^{-2} - \left(\frac{\mu}{\sigma^2}\frac{\cancel{n}-1}{n-3}\right)^2 \right\}$$

$$= \left(\frac{n-3}{n-1}\right)^2 \left\{ \left(\frac{\sigma^2}{n}+\mu^2\right) \frac{\Gamma\left(\frac{n-1}{2}-2\right)\left(\frac{2\sigma^2}{n-1}\right)^{-2}}{\Gamma\left(\frac{n-1}{2}\right)} - \frac{\mu^2(n-1)^2}{\sigma^4(n-3)^2} \right\}$$

Exercise 2

$$E\bar{X} = E\frac{1}{n}\sum X_i = \frac{1}{n}n\theta = \theta$$

$$E2\bar{Y} = \frac{2}{n}\sum EY_i = \frac{2}{n}\sum (EX_i)(EU_i) = \frac{2}{n}n\theta\frac{1}{2} = \theta$$

BOTH ARE UNBIASED.

$$VAR(2\bar{Y}) \equiv 4\,VAR\left(\frac{\sum Y_i}{n}\right) = \frac{4}{n^2}\sum\left[EY_i^2 - (EY_i)^2\right] = *$$

<u>NOTE</u>: $EY_i^2 = E X_i^2 U_i^2 = (1+\theta^2)\left(\frac{1}{12}+\frac{1}{2^2}\right) = \frac{1+\theta^2}{3}$

$EY_i = E X_i U_i = (EX_i)(EU_i) = \frac{\theta}{2}$

$$* = \frac{4}{n^2}n\left[\frac{1+\theta^2}{3} - \frac{\theta^2}{4}\right] = \frac{4+\theta^2}{3n}$$

$$VAR(\bar{X}) = \frac{1}{n}$$

RELATIVE EFFICIENCY OF $2\bar{Y}$ WITH RESPECT TO $\bar{X}$ IS:

$$\frac{\frac{4+\theta^2}{3n}}{\frac{1}{n}} = \frac{4+\theta^2}{3} > 1.$$

Exercise 3

FIND c SUCH THAT $E\hat{\theta} = \theta$

$$E\hat{\theta} = \frac{c}{n}\sum E(X_i^2)^{1/2}$$

MULTIPLY AND DIVIDE BY σ^2

TO GET $\frac{X_i^2}{\sigma^2} \sim \chi^2_1$ OR $\Gamma(\frac{1}{2},2)$

$$= \frac{c\sigma}{n}\sum E\left(\Gamma(\tfrac{1}{2},2)\right)^{1/2}$$

$$= \frac{c\sigma}{n}\, n\, \frac{\Gamma(\frac{1}{2}+\frac{1}{2})\cdot 2^{1/2}}{\Gamma(\frac{1}{2})} = \frac{c\sigma\sqrt{2}}{\sqrt{\pi}}.$$

NOW, $E\,\frac{c\sigma\sqrt{2}}{\sqrt{\pi}} = \sigma$, THEREFORE $c = \sqrt{\frac{\pi}{2}}$.

So $\hat{\theta} = \sqrt{\frac{\pi}{2}}\,\frac{1}{n}\sum (X_i^2)^{1/2} = \sigma\sqrt{\frac{\pi}{2}}\,\frac{1}{n}\sum\left(\frac{X_i^2}{\sigma^2}\right)^{1/2}$

$$\mathrm{VAR}\left(\frac{X_i^2}{\sigma^2}\right)^{1/2} = E\left[\Gamma(\tfrac{1}{2},2)\right] - \left(E\,\Gamma(\tfrac{1}{2},2)\right)^2$$

$$= \frac{\Gamma(\frac{1}{2}+2)\,2^2}{\Gamma(\frac{1}{2})} - \left(\sqrt{\frac{2}{\pi}}\right)^2 = 3 - \frac{2}{\pi} \rightarrow \mathrm{VAR}(\hat{\theta}) = \frac{\sigma^2}{2n^2}\, n\left(3-\frac{2}{\pi}\right)$$

THE MLE OF σ IS $\hat{\sigma} = \sqrt{\frac{1}{n}\sum X_i^2}$

THE CRAMER-RAO LOWER BOUND IS $\frac{\sigma^2}{2n}$

RELATIVE EFFICIENCY IS $3 - \frac{2}{\pi}$.

Exercise 4

(a)

$X_1, \ldots, X_m \overset{iid}{\sim} \exp(\lambda x)$

$Y_1, \ldots, Y_n \overset{iid}{\sim} \exp(\lambda y)$

$M_{X_i}(t) = \left(1 - \frac{t}{\lambda x}\right)^{-1}, \quad M_{Y_i}(t) = \left(1 - \frac{t}{\lambda y}\right)^{-1}$

$\sum X_i \sim \Gamma\left(m, \frac{1}{\lambda x}\right), \quad \sum Y_i \sim \Gamma\left(n, \frac{1}{\lambda y}\right)$

LET $\hat{\theta} = \frac{\sum Y_i}{\sum X_i}$ FIND $E\hat{\theta}$:

$E\hat{\theta} = \left(E \sum Y_i\right) \cdot \left(E\left(\sum X_i\right)^{-1}\right)$

$= \frac{n}{\lambda y} \cdot \frac{\Gamma(m-1)\left(\frac{1}{\lambda x}\right)^{-1}}{\Gamma(m)} = \frac{\lambda x}{\lambda y} \cdot \frac{n}{m-1}$

THEREFORE UNBIASED ESTIMATOR OF $\frac{\lambda x}{\lambda y}$

IS. $\hat{\theta}_1 = \frac{m-1}{n} \frac{\sum Y_i}{\sum X_i}$.

(b)

WE WANT TO MINIMIZE

$$MSE = E\left[\hat{\delta} - \frac{\lambda_x}{\lambda_y}\right]^2 = E\left[w\,\frac{\Sigma y_i}{\Sigma x_i} - \frac{\lambda_x}{\lambda_y}\right]^2 = \bigstar$$

<u>NOTE:</u> $c\,\dfrac{\bar{Y}}{\bar{X}} = w\,\dfrac{\Sigma y_i}{\Sigma x_i}$ WHERE $w = c\,\dfrac{m}{n}$

WE NEED $E(\Sigma y_i)^2$ AND $E(\Sigma x_i)^{-2}$

FROM (a): $\Sigma y_i \sim \Gamma\left(n, \frac{1}{\lambda_y}\right)$, $\Sigma x_i \sim \Gamma\left(m, \frac{1}{\lambda_x}\right)$

$$E(\Sigma y_i)^2 = VAR(\Sigma y_i) + (E\,\Sigma y_i)^2 = \frac{n}{\lambda_y^2} + \frac{n^2}{\lambda_y^2} = \frac{n(n+1)}{\lambda_y^2}$$

$$E(\Sigma x_i)^{-2} = \frac{\Gamma(m-2)\left(\frac{1}{\lambda_x}\right)^{-2}}{\Gamma(m)} = \frac{\Gamma(m-2)\left(\frac{1}{\lambda_x^2}\right)^{-2}}{(m-1)(m-2)\Gamma(m-2)} = \frac{\lambda_x^2}{(m-1)(m-2)}$$

$$\bigstar = w^2 E\frac{(\Sigma y_i)^2}{(\Sigma x_i)^2} - 2w\frac{\lambda_x}{\lambda_y}E\frac{\Sigma y_i}{\Sigma x_i} + \frac{\lambda_x^2}{\lambda_y^2}$$

$$= w^2\frac{\lambda_x^2}{\lambda_y^2}\frac{n(n+1)}{(m-1)(m-2)} - 2w\frac{\lambda_x^2}{\lambda_y^2}\frac{n}{m-1} + \frac{\lambda_x^2}{\lambda_y^2}$$

THIS MINIMIZED WHEN $w = \dfrac{m-2}{n+1}$

AND THEREFORE $\delta = \dfrac{n(m-2)}{m(n+1)}$

EXERCISE 5:

$$P\left(X_1=x_1, \ldots, X_n=x_n \mid U=u\right) = \frac{P\left(X_1=x_1, \ldots, X_n=x_n\right)}{P(U=u)} = *$$

SINCE $U=\sum X_i$ IT FOLLOWS THAT $U \sim \text{POISSON}(n\lambda)$

$$* = \frac{\frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \cdots \frac{\lambda^{x_n} e^{-\lambda}}{x_n!}}{\frac{(n\lambda)^u e^{-n\lambda}}{u!}} = \frac{n!}{x_1! \cdots x_n!} \frac{\lambda^{\sum x_i} e^{-n\lambda}}{n^u \lambda^u e^{-n\lambda}}$$

$$= \frac{n!}{x_1! \cdots x_n!} \frac{1}{n^{\sum x_i}} \quad \left(\text{BECAUSE } u=\sum x_i\right).$$

WE SEE THAT THIS EXPRESSION IS FREE OF THE PARAMETER λ THEREFORE $U=\sum X_i$ IS A SUFFICIENT STATISTIC FOR λ.

EXERCISE 6 :

$$f(x) = \frac{2x}{\theta} e^{-\frac{x^2}{\theta}} , \quad x > 0$$

$$L = \frac{2^n \prod x_i}{\theta^n} e^{-\frac{1}{\theta} \sum x_i^2}$$

$$L = \frac{e^{-\frac{1}{\theta} \sum x_i^2}}{\theta^n} \; 2^n \prod x_i$$

$$L = g(u, \theta) \cdot h(x)$$

WHERE $g(u, \theta) = \dfrac{e^{-\frac{1}{\theta} \sum x_i^2}}{\theta^n}$

AND $h(x) = 2^n \prod x_i$

THEREFORE, $U = \sum x_i^2$
IS A SUFFICIENT STATISTIC FOR θ.