Answer the following questions:

a. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. \( \text{Poisson}(\lambda) \) and let \( \bar{X} \) and \( S^2 \) be the sample mean and sample variance respectively. Each one of these two estimators has expected value equal to \( \lambda \) (why?). Which estimator is better? Hint: The answer should be: “\( \bar{X} \) is at least as good as \( S^2 \)”. Explain why by finding the Rao-Cramér lower bound of an unbiased estimator of \( \lambda \).

b. Let \( X_1, \ldots, X_n \) be i.i.d. \( \text{N}(\theta, \theta) \), \( \theta > 0 \). For this model both \( \bar{X} \) and \( cS \) are unbiased estimators of \( \theta \), where \( c = \frac{\sqrt{n-1} \Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2 \Gamma\left(\frac{n}{2}\right)}} \). Show that for any \( \alpha \) the estimator \( \alpha \bar{X} + (1 - \alpha)cS \) is also unbiased estimator of \( \theta \). For what value of \( \alpha \) this estimator has the minimum variance?

c. Let \( X_1, \ldots, X_n \) be i.i.d. random variables with \( X_i \sim \text{Gamma}(\alpha, \beta) \) with \( \alpha \) known. Find an unbiased estimator of \( \frac{1}{\beta} \). (Find \( E\bar{X} \) and then adjust it to be unbiased of \( \frac{1}{\beta} \).)

d. Let \( X_1, \ldots, X_n \) be i.i.d. random variables with \( X_i \sim \text{Gamma}(\alpha, \beta) \) with \( \alpha \) known. Is \( \hat{\beta} = \frac{\bar{X}}{\alpha} \) efficient estimator of \( \beta \)?