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Statistics 100B **Instructor: Nicolas Christou**

Hypothesis testing

- Elements of a hypothesis test:
  1. Null hypothesis, H_0 (claim about $\mu, p, \sigma^2, \mu_1 - \mu_2$, etc. It can be $\leq, \geq, =$).
  2. Alternative hypothesis, H_a ($>, <, \neq$).
  3. Test statistic.
  4. Significance level α.

- Hypothesis testing for μ:
  $H_0 : \mu = \mu_0$
  $H_a : \mu > \mu_0$, $\mu < \mu_0$, $\mu \neq \mu_0$ (use only one of these!)

- When σ is known:
  Test statistic
  $$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

- When σ is unknown:
  Test statistic
  $$t = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$$

- If σ is known:
  Reject H_0 if Z falls in the rejection region. The rejection region is based on the significance level α we choose.

- If σ is unknown:
  Reject H_0 if t falls in the rejection region. The rejection region is based on the significance level α we choose and the degrees of freedom $n-1$.

- The $p-$value of a test. It is the probability of seeing the test statistic or a more extreme value (extreme is towards the direction of the alternative). If $p-$value $< \alpha$ then H_0 is rejected. This is another way of testing a hypothesis (it should always agree with testing using Z or t).

Hypothesis Test For Population Mean μ

Hypothesis Test When σ Is Known:

$H_0: \ \mu = \mu_0$

Alternative Hypothesis H_a	Reject H_0 If
$\mu < \mu_0$	$Z < -Z_\alpha$
$\mu > \mu_0$	$Z > \ Z_\alpha$
$\mu \neq \mu_0$	$Z < -Z_{\alpha/2}$ or $Z > \ Z_{\alpha/2}$

$$Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$$

Hypothesis Test When σ Is Not Known:

$H_0: \ \mu = \mu_0$

Alternative Hypothesis H_a	Reject H_0 If
$\mu < \mu_0$	$t < -t_{\alpha;n-1}$
$\mu > \mu_0$	$t > \ t_{\alpha;n-1}$
$\mu \neq \mu_0$	$t < -t_{\alpha/2;n-1}$ or $t > \ t_{\alpha/2;n-1}$

$$t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$$

Hypothesis Test For Proportion:

$H_0: \ p = p_0$

Alternative Hypothesis H_a	Reject H_0 If
$p < p_0$	$Z < -Z_\alpha$
$p > p_0$	$Z > \ Z_\alpha$
$p \neq p_0$	$Z < -Z_{\alpha/2}$ or $Z > \ Z_{\alpha/2}$

$$Z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Hypothesis testing - examples

Example 1

A manufacturer of chocolates claims that the mean weight of a certain box of chocolates is 368 grams. The standard deviation of the box's weight is known to be $\sigma = 10$ grams. If a sample of 49 boxes has sample mean $\overline{x} = 364$ grams, test the hypothesis that the mean weight of the boxes is less than 368 grams. Use $\alpha = 0.05$ level of significance.

Solution:

1.

$$\begin{aligned} H_0: &\quad \mu = 368 \\ H_a: &\quad \mu < 368 \end{aligned}$$

2. We compute the test statistic z:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{364 - 368}{\frac{10}{\sqrt{49}}} \Rightarrow z = -2.8$$

3. We find the rejection region. Here we use significance level $\alpha = 0.05$, therefore the rejection region is when $z < -1.645$.

4. Conclusion: Since $z = -2.8 < -1.645$ we reject H_0.

Compute the $p-$value of the test:

$$p - \text{value} = P(\bar{X} < 364) = P(Z < -2.8) = 0.0026.$$

Rule: If $p-$value $< \alpha$ then H_0 is rejected. Again, using the $p-$value we reject H_0.

Example 2

A large retailer wants to determine whether the mean income of families living whithin 2 miles of a proposed building site exceeds \$24400. What can we conclude at the 0.05 level of significance if the sample mean income of 60 families is $\overline{x} = \$24524$? Use $\sigma = \$763$.

Solution:

1.

$$\begin{aligned} H_0: &\quad \mu = 24400 \\ H_a: &\quad \mu > 24400 \end{aligned}$$

2. We compute the test statistic z:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{24524 - 24400}{\frac{763}{\sqrt{60}}} \Rightarrow z = 1.26$$

3. We find the rejection region. Here we use significance level $\alpha = 0.05$, therefore the rejection region is when $z > 1.645$.

4. Conclusion: Since $z = 1.26$ does not fall in the R.R. we do not reject H_0.

Example 3
It is claimed that the mean mileage of a certain type of vehicle is 35 miles per gallon of gasoline with population standard deviation $\sigma = 5$ miles. What can be concluded using $\alpha = 0.01$ about the claim if a random sample of 49 such vehicles has sample mean $\overline{x} = 36$ miles?

Solution:

1.

$$\begin{aligned} H_0: &\quad \mu = 35 \\ H_a: &\quad \mu \neq 35 \end{aligned}$$

2. We compute the test statistic z:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{36 - 35}{\frac{5}{\sqrt{49}}} \Rightarrow z = 1.4$$

3. We find the rejection region. Here we use significance level $\alpha = 0.01$, but because of a two-sided test we have two rejection regions. They are $z < -2.575$ or $z > 2.575$.

4. Conclusion: Since $z = 1.4$ does not fall in any of the two rejection regions we do not reject H_0.

When we have a two-sided test the $p-$value is computed as follows:

$$p - \text{value} = 2P(\bar{X} > 36) = 2P(Z > 1.4) = 2(1 - 0.9192) = 0.1616.$$

Again, using the $p-$value H_0 is not rejected.

Example 4
A manufacturer claims that 20% of the public preferred her product. A sample of 100 persons is taken to check her claim. It is found that 8 of these 100 persons preferred her product.

a. Find the p-value of the test (use a two-tailed test).

b. Using the 0.05 level of significance test her claim.

Solution:
We test the following hypothesis:

$$\begin{aligned} H_0: &\quad p = 0.20 \\ H_a: &\quad p \neq 0.20 \end{aligned}$$

We compute the test statistic z:

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.08 - 0.20}{\sqrt{\frac{0.20(1-0.20)}{100}}} = -3.0.$$

Therefore the $p-$value is:

$$p - \text{value} = 2P(\hat{p} < 0.08) = 2P(Z < -3.0) = 2(0.0013) = 0.0026.$$

We reject H_0 because $p-$value$= 0.0026 < 0.05$.

Hypothesis testing - t distribution

Example 1
A tire manufacturer hopes that their newly designed tires will allow a car traveling at 60 mph to come to a complete stop within an average of 125 feet after the brakes are applied. They will adopt the new tires unless there is strong evidence that the tires do not meet this objective. The distances (in feet) for 9 stops on a test track were 129, 128, 130, 132, 135, 123, 125, 128, and 130. These data have $\bar{x} = 128.89, s = 3.55$. Test an appropriate hypothesis to conclude whether the company should adopt the new tires. Use $\alpha = 0.05$.

Solution:

1.
$$H_0: \quad \mu = 125$$
$$H_a: \quad \mu > 125$$

2. We compute the test statistic t:
$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{128.89 - 125}{\frac{3.55}{\sqrt{9}}} \Rightarrow t = 3.29.$$

3. We find the rejection region. Here we use significance level $\alpha = 0.05$ with $n - 1 = 9 - 1 = 8$ degrees of freedom. Therefore the rejection region is when $t > 1.860$.

4. Conclusion: Since $t = 3.29$ falls in any the rejection region we reject H_0.

The $p-$value is: $p-$value$= P(\bar{X} > 128.89) = P(t > 3.29)$. From the t table we can say that the $0.005 < p-$value < 0.01 Again, using the $p-$value H_0 is rejected.

Example 2 (from *Mathematical Statistics and Data Analysis*), by J. Rice, 2nd Edition.
In a study done at the National Institute of Science and Technology (Steel et al. 1980), asbestos fibers on filters were counted as part of a project to develop measurement standards for asbestos concentration. Asbestos dissolved in water was spread on a filter, and punches of 3-mm diameter were taken from the filter and mounted on a transmission electron microscope. An operator counted the number of fibers in each of 23 grid squares, yielding the following counts:

31	29	19	18	31	28
34	27	34	30	16	18
26	27	27	18	24	22
28	24	21	17	24	

Assume normal distribution. These data have $\bar{x} = 24.91, s = 5.48$. Using $\alpha = 0.05$ test the following hypothesis:
$H_0: \mu = 18$
$H_a: \mu \neq 18$

Solution:
We compute the test statistic t:
$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{24.91 - 18}{\frac{5.48}{\sqrt{23}}} \Rightarrow t = 6.05$$

We find the rejection region. Here we use significance level $\alpha = 0.05$ with $n - 1 = 23 - 1 = 22$ degrees of freedom. Therefore the rejection region is when $t < -2.074$ or $t > 2.074$. Conclusion: Since $t = 6.05$ falls in one of the rejection regions we reject H_0.

Compute the $p-$value of the test: This is a two-sided test therefore the $p-$value is
$p-$value$= 2P(\bar{X} > 24.91) = 2P(t > 6.05)$. From the t table we can only say that $p-$value is less that 0.01.

More examples - Hypothesis testing

Example 1

An experimenter has prepared a drug dosage level that he claims will induce sleep for at least 80% of those people suffering from insomnia. After examining the dosage, we feel that his claims regarding the efectiveness of the dosage are inflated. In an attempt to disprove his claim, we administer his prescribed dosage to 20 insomniancs, and we observe X, the number having sleep induced by the drug dose. We wish to test the hypothesis $H_0 : p = 0.8$ against the alternative $H_a : p < 0.8$. Assume the rejection region $X \leq 12$ is used.

a. Find the type I error α.

b. Find the type II error β if the true $p = 0.6$.

c. Find the type II error β if the true $p = 0.4$.

Example 2

For a certain candidate's political poll $n = 15$ voters are sampled. Assume that this sample is taken from an infinite population of voters. We wish to test H_0: $p = 0.5$ against the alternative H_a: $p < 0.5$. The test statistic is X, which is the number of voters among the 15 sampled favoring this candidate.

a. Calculate the probability of a type I error α if we select the rejection region to be $RR = \{x \leq 2\}$.

b. Is our test good in protecting us from concluding that this candidate is a winner if, in fact, he will lose? Suppose that he really will win 30% of the vote ($p = 0.30$). What is the probability of a type II error β that the sample will erroneously lead us to conclude that H_0 is true?

Comparison between confidence intervals and a two-tailed hypothesis test

Two dice are rolled and the sum X of the two numbers that occured is recorded. The probability distribution of X is as follows:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

This distribution has mean $\mu = 7$ and standard deviation $\sigma = 2.42$. We take 100 samples of size $n = 50$ each from this distribution and compute for each sample the sample mean $\bar{x}$. Pretend now that we only know that $\sigma = 2.42$, and that μ is unknown. We are going to use these 100 sample means to construct 100 95% confidence intervals for the true population mean μ, and to test using level of significance $\alpha = 0.05$ 100 times the hypothesis:

$H_0 : \mu = 7$

$H_a : \mu \neq 7$

The results are as follows:

Sample	$\bar{x}$	95% C.I. for μ	Is $\mu = 7$ included?	$z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$	Reject H_0?
1	6.9	$6.23 \leq \mu \leq 7.57$	YES	-0.29	NO
2	6.3	$5.63 \leq \mu \leq 6.97$	NO	-2.05	YES
3	6.58	$5.91 \leq \mu \leq 7.25$	YES	-1.23	NO
4	6.54	$5.87 \leq \mu \leq 7.21$	YES	-1.34	NO
5	6.7	$6.03 \leq \mu \leq 7.37$	YES	-0.88	NO
6	6.58	$5.91 \leq \mu \leq 7.25$	YES	-1.23	NO
7	7.2	$6.53 \leq \mu \leq 7.87$	YES	0.58	NO
8	7.62	$6.95 \leq \mu \leq 8.29$	YES	1.81	NO
9	6.94	$6.27 \leq \mu \leq 7.61$	YES	-0.18	NO
10	7.36	$6.69 \leq \mu \leq 8.03$	YES	1.05	NO
11	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
12	7.08	$6.41 \leq \mu \leq 7.75$	YES	0.23	NO
13	7.42	$6.75 \leq \mu \leq 8.09$	YES	1.23	NO
14	7.42	$6.75 \leq \mu \leq 8.09$	YES	1.23	NO
15	6.8	$6.13 \leq \mu \leq 7.47$	YES	-0.58	NO
16	6.94	$6.27 \leq \mu \leq 7.61$	YES	-0.18	NO
17	7.2	$6.53 \leq \mu \leq 7.87$	YES	0.58	NO
18	6.7	$6.03 \leq \mu \leq 7.37$	YES	-0.88	NO
19	7.1	$6.43 \leq \mu \leq 7.77$	YES	0.29	NO
20	7.04	$6.37 \leq \mu \leq 7.71$	YES	0.12	NO
21	6.98	$6.31 \leq \mu \leq 7.65$	YES	-0.06	NO
22	7.18	$6.51 \leq \mu \leq 7.85$	YES	0.53	NO
23	6.8	$6.13 \leq \mu \leq 7.47$	YES	-0.58	NO
24	6.94	$6.27 \leq \mu \leq 7.61$	YES	-0.18	NO
25	8.1	$7.43 \leq \mu \leq 8.77$	NO	3.21	YES
26	7	$6.33 \leq \mu \leq 7.67$	YES	0.00	NO
27	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
28	6.82	$6.15 \leq \mu \leq 7.49$	YES	-0.53	NO
29	6.96	$6.29 \leq \mu \leq 7.63$	YES	-0.12	NO
30	7.46	$6.79 \leq \mu \leq 8.13$	YES	1.34	NO
31	7.04	$6.37 \leq \mu \leq 7.71$	YES	0.12	NO
32	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
33	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
34	6.8	$6.13 \leq \mu \leq 7.47$	YES	-0.58	NO
35	7.12	$6.45 \leq \mu \leq 7.79$	YES	0.35	NO
36	7.18	$6.51 \leq \mu \leq 7.85$	YES	0.53	NO
37	7.08	$6.41 \leq \mu \leq 7.75$	YES	0.23	NO
38	7.24	$6.57 \leq \mu \leq 7.91$	YES	0.70	NO
39	6.82	$6.15 \leq \mu \leq 7.49$	YES	-0.53	NO
40	7.26	$6.59 \leq \mu \leq 7.93$	YES	0.76	NO
41	7.34	$6.67 \leq \mu \leq 8.01$	YES	0.99	NO
42	6.62	$5.95 \leq \mu \leq 7.29$	YES	-1.11	NO
43	7.1	$6.43 \leq \mu \leq 7.77$	YES	0.29	NO
44	6.98	$6.31 \leq \mu \leq 7.65$	YES	-0.06	NO
45	6.98	$6.31 \leq \mu \leq 7.65$	YES	-0.06	NO
46	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
47	7.14	$6.47 \leq \mu \leq 7.81$	YES	0.41	NO
48	7.5	$6.83 \leq \mu \leq 8.17$	YES	1.46	NO
49	7.08	$6.41 \leq \mu \leq 7.75$	YES	0.23	NO
50	7.32	$6.65 \leq \mu \leq 7.99$	YES	0.94	NO

Sample	$\bar{x}$	$95\% C.I. for \mu$	Is $\mu = 7$ included?	$z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$	Reject H_0?
51	6.54	$5.87 \leq \mu \leq 7.21$	YES	-1.34	NO
52	7.14	$6.47 \leq \mu \leq 7.81$	YES	0.41	NO
53	6.64	$5.97 \leq \mu \leq 7.31$	YES	-1.05	NO
54	7.46	$6.79 \leq \mu \leq 8.13$	YES	1.34	NO
55	7.34	$6.67 \leq \mu \leq 8.01$	YES	0.99	NO
56	7.28	$6.61 \leq \mu \leq 7.95$	YES	0.82	NO
57	6.56	$5.89 \leq \mu \leq 7.23$	YES	-1.29	NO
58	7.72	$7.05 \leq \mu \leq 8.39$	NO	2.10	YES
59	6.66	$5.99 \leq \mu \leq 7.33$	YES	-0.99	NO
60	6.8	$6.13 \leq \mu \leq 7.47$	YES	-0.58	NO
61	7.08	$6.41 \leq \mu \leq 7.75$	YES	0.23	NO
62	6.58	$5.91 \leq \mu \leq 7.25$	YES	-1.23	NO
63	7.3	$6.63 \leq \mu \leq 7.97$	YES	0.88	NO
64	7.1	$6.43 \leq \mu \leq 7.77$	YES	0.29	NO
65	6.68	$6.01 \leq \mu \leq 7.35$	YES	-0.94	NO
66	6.98	$6.31 \leq \mu \leq 7.65$	YES	-0.06	NO
67	6.94	$6.27 \leq \mu \leq 7.61$	YES	-0.18	NO
68	6.78	$6.11 \leq \mu \leq 7.45$	YES	-0.64	NO
69	7.2	$6.53 \leq \mu \leq 7.87$	YES	0.58	NO
70	6.9	$6.23 \leq \mu \leq 7.57$	YES	-0.29	NO
71	6.42	$5.75 \leq \mu \leq 7.09$	YES	-1.69	NO
72	6.48	$5.81 \leq \mu \leq 7.15$	YES	-1.52	NO
73	7.12	$6.45 \leq \mu \leq 7.79$	YES	0.35	NO
74	6.9	$6.23 \leq \mu \leq 7.57$	YES	-0.29	NO
75	7.24	$6.57 \leq \mu \leq 7.91$	YES	0.70	NO
76	6.6	$5.93 \leq \mu \leq 7.27$	YES	-1.17	NO
77	7.28	$6.61 \leq \mu \leq 7.95$	YES	0.82	NO
78	7.18	$6.51 \leq \mu \leq 7.85$	YES	0.53	NO
79	6.76	$6.09 \leq \mu \leq 7.43$	YES	-0.70	NO
80	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
81	7	$6.33 \leq \mu \leq 7.67$	YES	0.00	NO
82	7.08	$6.41 \leq \mu \leq 7.75$	YES	0.23	NO
83	7.18	$6.51 \leq \mu \leq 7.85$	YES	0.53	NO
84	7.26	$6.59 \leq \mu \leq 7.93$	YES	0.76	NO
85	6.88	$6.21 \leq \mu \leq 7.55$	YES	-0.35	NO
86	6.28	$5.61 \leq \mu \leq 6.95$	NO	-2.10	YES
87	7.06	$6.39 \leq \mu \leq 7.73$	YES	0.18	NO
88	6.66	$5.99 \leq \mu \leq 7.33$	YES	-0.99	NO
89	7.18	$6.51 \leq \mu \leq 7.85$	YES	0.53	NO
90	6.86	$6.19 \leq \mu \leq 7.53$	YES	-0.41	NO
91	6.96	$6.29 \leq \mu \leq 7.63$	YES	-0.12	NO
92	7.26	$6.59 \leq \mu \leq 7.93$	YES	0.76	NO
93	6.68	$6.01 \leq \mu \leq 7.35$	YES	-0.94	NO
94	6.76	$6.09 \leq \mu \leq 7.43$	YES	-0.70	NO
95	7.3	$6.63 \leq \mu \leq 7.97$	YES	0.88	NO
96	7.04	$6.37 \leq \mu \leq 7.71$	YES	0.12	NO
97	7.34	$6.67 \leq \mu \leq 8.01$	YES	0.99	NO
98	6.72	$6.05 \leq \mu \leq 7.39$	YES	-0.82	NO
99	6.64	$5.97 \leq \mu \leq 7.31$	YES	-1.05	NO
100	7.3	$6.63 \leq \mu \leq 7.97$	YES	0.88	NO

Hypothesis Testing - Type I and Type II error

		ACTUAL SITUATION	
		H_0 IS TRUE	H_0 IS NOT TRUE
STATISTICAL DECISION	DO NOT REJECT H_0	Correct Decision $1-\alpha$	Type II error β
	REJECT H_0	Type I Error α	Correct Decision $1-\beta$ (Power)

Type II error (β) and the power of the test ($1-\beta$)

Example:
A manufacturer of tires claims that the mean lifetime of these tires is 25000 miles. A random sample of 100 tires will be selected, and assume that the population standard deviation is 3500 miles. Calculate the probability of a Type II error (β) and the power of the test ($1-\beta$) if the true population mean is 23500 miles using:

a. $\alpha = 0.05$

b. $\alpha = 0.01$

Solution:

a. $\alpha = 0.05$.
We are testing the hypothesis
$H_0 : \mu = 25000$
$H_a : \mu < 25000$

We find first the values of $\bar{X}$ for which H_0 is rejected. H_0 is rejected when $Z < -1.645$. Or

$$\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < -1.645$$
$$\frac{\bar{X}-25000}{\frac{3500}{\sqrt{100}}} < -1.645$$

Therefore

$$\bar{X} < 25000 - 1.645\frac{3500}{\sqrt{100}} \Rightarrow \bar{X} < 24424.25$$

If the sample of size $n = 100$ gives a value of $\bar{X}$ less than 24424.25 then H_0 s rejected.

How do we compute the Type II error β?

$$\begin{aligned}\beta &= P(\text{falsely accepting } H_0) = P(\bar{X} > 24424.25, \text{when } \mu = 23500) \\ &= P\left(Z > \frac{24424.25-23500}{\frac{3500}{\sqrt{100}}}\right) = P(Z > 2.64) = 1 - 0.9959 \Rightarrow \beta = 0.0041.\end{aligned}$$

Therefore, the power of the test is $1-\beta = 0.9959$.

b. $\alpha = 0.01$
We reject H_0 if $Z < -2.325$ or $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} < -1.645$ or $\bar{X} < 24186.25$.

$$\begin{aligned}\beta &= P(\text{falsely accepting } H_0) = P(\bar{X} > 24186.25, \text{when } \mu = 23500) \\ &= P\left(Z > \frac{24186.25-23500}{\frac{3500}{\sqrt{100}}}\right) = P(Z > 1.96) = 1 - 0.9750 \Rightarrow \beta = 0.025.\end{aligned}$$

Therefore, the power of the test is $1-\beta = 0.9750$.

Sample size determination
Type I and Type II errors are given

A manufacturer of tires claims that the mean lifetime of these tires is at least 25000 miles when the production process is working properly. Based upon past experience, the standard deviation of the lifetime of the tires is 3500 miles. The production manager will stop the production if there is evidence that the mean lifetime of the tires is below 25000 miles.

a. If the production manager wishes to have 80% power of detecting a shift in the lifetime mean of the tires from 25000 to 24000 miles and if he is willing to take a 5% risk of committing a Type I error, what sample size must be selected?

b. If the production manager wishes to have 80% power of detecting a shift in the lifetime mean of the tires from 25000 to 23000 miles and if he is willing to take a 5% risk of committing a Type I error, what sample size must be selected?

Power of a test when $H_a : \mu < \mu_0$

Suppose that we want to determine whether or not a cereal box packaging process is in control. The process is in control if the mean weight of a box is at least 368 grams. Therefore we would be interested in testing whether the mean weight is less than 368 grams. The two hypotheses are formulated as below:

$H_0 : \mu \geq 368$

$H_a : \mu < 368$

Let's assume that that the standard deviation of the filling process is known to be $\sigma = 15$ grams and that the weight of the box follows the normal distribution. To test this hypothesis a sample of $n = 25$ boxes of cereal is to be selected. Our goal here is to find the power of the test for different true values of μ when we are willing to take a risk of Type I error $\alpha = 0.05$. In other words we want to compute the power of the test when there is a shift from $\mu = 368$ grams to μ_a, when $\mu_a < 368$. The table below gives the power of the test for different values of μ_a.

μ_a	Power $(1-\beta)$
352	0.9999
353	0.9996
354	0.9987
355	0.9964
356	0.9906
357	0.9783
358	0.9545
359	0.9115
360	0.8461
361	0.7549
362	0.6368
363	0.5080
364	0.3783
365	0.2578
366	0.1635
367	0.0951
368	0.0500

This is how the power was computed for this example:

$1 - \beta =$P(reject H_0 when H_0 is not true). In our example we will reject H_0 (conclude that $\mu < \mu_0$ where μ_0 is the value of μ under the null hypothesis) if

$z < -z_\alpha \Rightarrow \frac{\bar{x}-\mu 0}{\frac{\sigma}{\sqrt{n}}} < -z_\alpha \Rightarrow \bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$. For these values of $\bar{x}$ we reject H_0. Now the power is the probability of finding a value of $\bar{x}$ in the above range when the true mean is μ_a. This is $1 - \beta = P(\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} | \mu = \mu_a) = P(z < \frac{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}})$.

Power of a test when $H_a : \mu > \mu_0$

Suppose that we want to determine whether or not a cereal box packaging process is in control. The process is in control if the mean weight of a box is at most 368 grams. Therefore we would be interested in testing whether the mean weight is more than 368 grams. The two hypotheses are formulated as below:

$H_0 : \mu \leq 368$

$H_a : \mu > 368$

Let's assume that that the standard deviation of the filling process is known to be $\sigma = 15$ grams and that the weight of the box follows the normal distribution. To test this hypothesis a sample of $n = 25$ boxes of cereal is to be selected. Our goal here is to find the power of the test for different true values of μ when we are willing to take a risk of Type I error $\alpha = 0.05$. In other words we want to compute the power of the test when there is a shift from $\mu = 368$ grams to μ_a, when $\mu_a > 368$. The table below gives the power of the test for different values of μ_a.

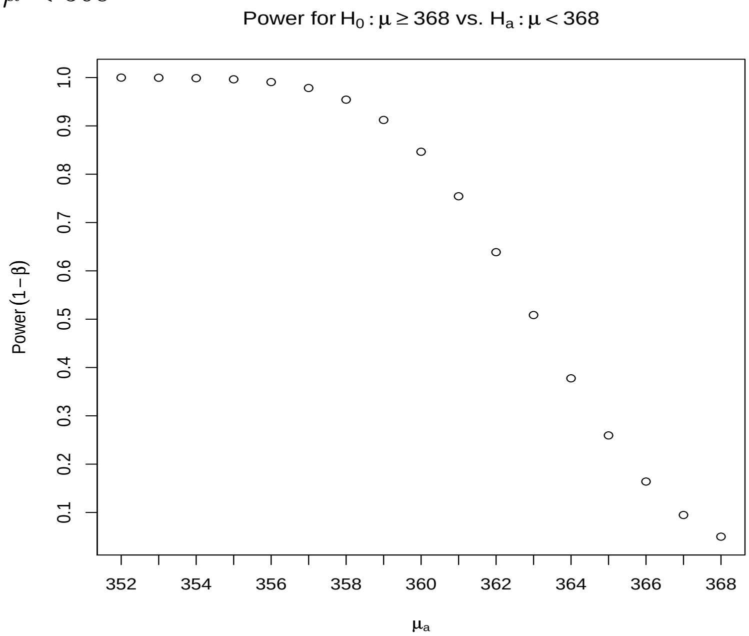
μ_a	Power $(1-\beta)$
368	0.0500
369	0.0951
370	0.1635
371	0.2578
372	0.3783
373	0.5080
374	0.6368
375	0.7549
376	0.8461
377	0.9115
378	0.9545
379	0.9783
380	0.9906
381	0.9964
382	0.9987
383	0.9996
384	0.9999

The power of the test $(1-\beta)$ here is computed as follows:

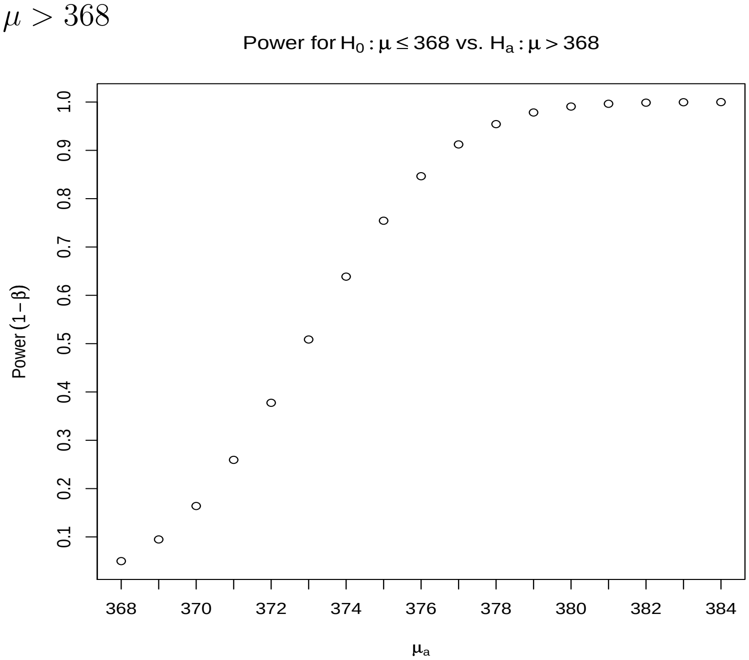
$1-\beta =$P(reject H_0 when H_0 is not true). In our example we will reject H_0 (conclude that $\mu > \mu_0$ where μ_0 is the value of μ under the null hypothesis) if $z > z_\alpha \Rightarrow \frac{\bar{x}-\mu 0}{\frac{\sigma}{\sqrt{n}}} < z_\alpha \Rightarrow \bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$. For these values of $\bar{x}$ we reject H_0. Now the power is the probability of finding a value of $\bar{x}$ in the above range when the true mean is μ_a. This is $1-\beta = P(\bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} | \mu = \mu_a) = P(z > \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}})$.

Power curves:
The graph of the power against μ_a for the two examples above is shown below.

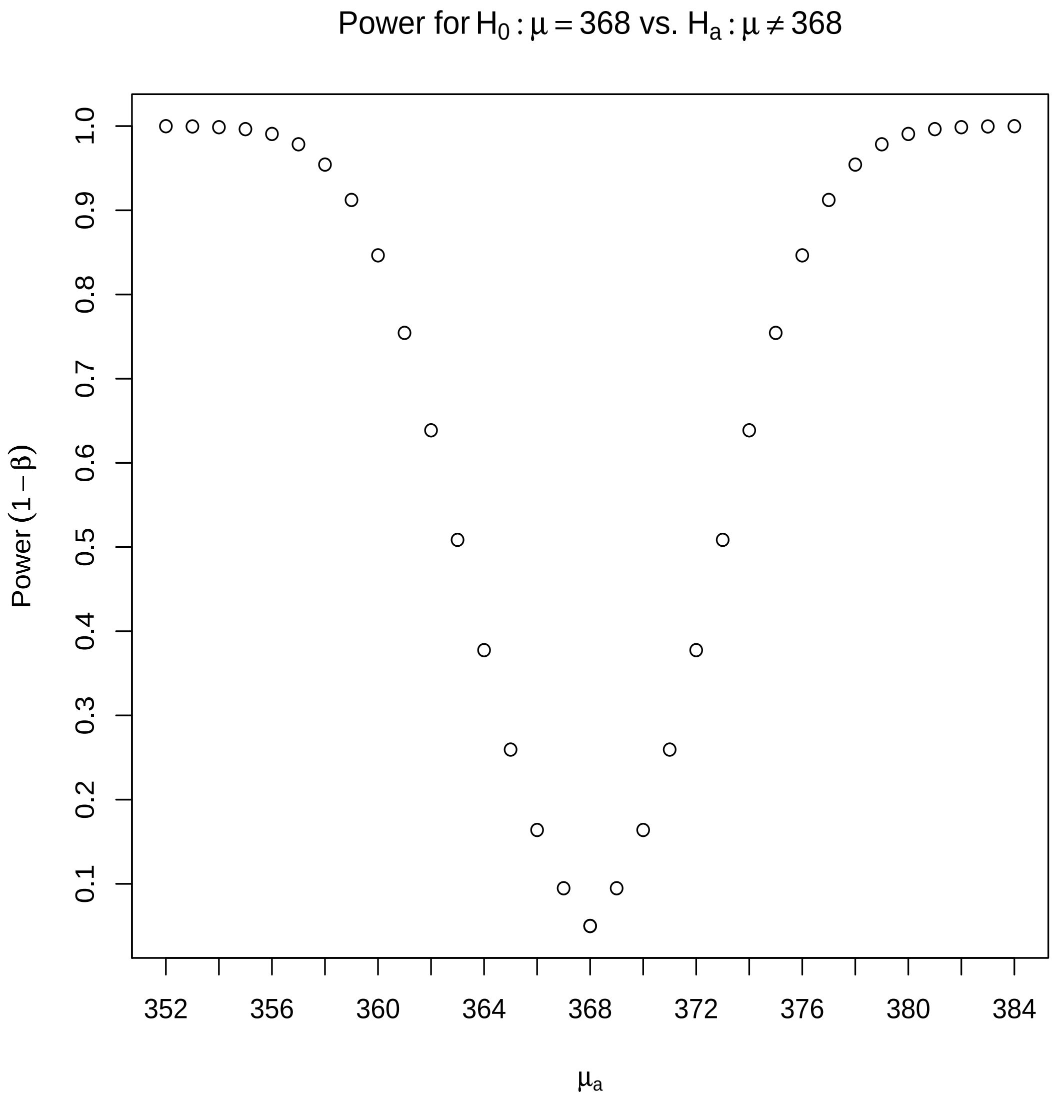
$H_0 : \mu \geq 368$
$H_a : \mu < 368$



$H_0 : \mu \leq 368$
$H_a : \mu > 368$



If we combine the two cases above we will get the graph of the power for a two-sided test.
$H_0 : \mu = 368$
$H_a : \mu \neq 368$



Other hypothesis tests

Test for the difference between two population means:
$H_0 : \mu_1 - \mu_2 = \delta$ (δ could be 0, and the test is whether $\mu_1 = \mu_2$).
$H_a : \mu_1 - \mu_2 > \delta$, or $\mu_1 - \mu_2 < \delta$, or $\mu_1 - \mu_2 \neq \delta$

In order to test this hypothesis we select two samples from the two populations. Let the two samples be $X_1, X_2, \cdots, X_n$, and $Y_1, Y_2, \cdots, Y_n$. The test statistic is based on the difference of the two sample means, $\bar{X} - \bar{Y}$, and it depends on whether σ_1^2, σ_2^2 are known, whether $\sigma_1^2 = \sigma_2^2$, whether the sample sizes are small or large. Below we summarize all these different cases.

a. The two variances, σ_1^2, σ_2^2 are known, and the two populations are normal. Then regardless of the size of the two samples (could be small or large), the test statistics is:

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

If Z falls in the rejection region (based on the significance level α) then H_0 is rejected.

b. The two variances are unknown and $n_1 \geq 30, n_2 \geq 30$, (large samples). We will estimate the two unknown variances with the sample variances, s_1^2, s_2^2. Because the two samples are large we can still use the Z test as an approximation.

$$Z \approx \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

If Z falls in the rejection region (based on the significance level α) then H_0 is rejected.

c. The two variances are unknown but equal ($\sigma_1^2 = \sigma_2^2$) and $n_1 \leq 30$, or $n_2 \leq 30$, (one or both of the samples are small). We will estimate the unknown but common variance with the so called pooled variance s^2_{pooled}, and the test statistic will be t with n_1+n_2-2 degrees of freedom.

$$t = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{s^2_{\text{pooled}}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$s^2_{\text{pooled}} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

If t falls in the rejection region (based on the significance level α and $df = n_1 + n_2 - 2$) then H_0 is rejected.