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Statistics 100B **Instructor: Nicolas Christou**

Hypothesis testing

- A hypothesis test is a claim about a parameter of a population.

- Given the data we want to make a decision about which of two hypothesis is true (or not true).

- The two hypotheses are called the "null" and "alternative" hypotheses (denoted with H_0 and H_a respectively).

- The test has the following formulation:
  $H_0: \theta \in \Theta_0$
  $H_a: \theta \in \Theta_0'$, where Θ_0' is the complement of Θ_0.

- Examples:

  1. Consider the simple regression model: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$. We wish to test
     $H_0: \beta_1 = 0$ (the null hypothesis states that there is no association between the response y and the predictor x).
     $H_a: \beta_1 \neq 0$ (the alternative hypothesis states that there is a linear association between y and x).

  2. Consider an experiment in which a patient is given a treatment (some drug) and we want to test if there is a difference between before and after administrating the drug. We wish to test
     $H_0: \mu_d = 0$ (the null hypothesis states that there is no difference).
     $H_a: \mu_d \neq 0$ (the alternative hypothesis states that there is a difference).

  3. Consider an experiment where the goal is to see if on average there is a difference in the production of corn using different fertilizers. We wish to test
     $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$ (the null hypothesis states that the production is the same under the different fertilizers (treatments)).
     H_a : At least two means are not equal (the alternative hypothesis states that there are differences).

  4. Test for the proportion of defective items at a certain production line:
     $H_0: p = p_0$
     $H_a: p > p_0$.

- We need to find and evaluate hypothesis tests.
- Find a procedure that will tell us for which sample values H_0 is accepted (and therefore for which sample values H_0 is rejected). These are called the acceptance region (accepts H_0) and the rejection region (rejects H_0).
- Usually the procedure of rejecting (or accepting) involves the so called test statistic $T(\mathbf{X})$ which a function of the data $\mathbf{X} = (X_1, \ldots, X_n)'$.
- Type I and Type II error

		ACTUAL SITUATION	
		H_0 IS TRUE	H_0 IS NOT TRUE
STATISTICAL DECISION	DO NOT REJECT H_0	Correct Decision $1-\alpha$	Type II error β
	REJECT H_0	Type I Error α	Correct Decision $1-\beta$ (Power)

- Testing a simple hypothesis, i.e. $H_0 : \theta = \theta'$ against $H_a : \theta = \theta''$.

1. Best critical region of size α.
   Definition: Let R denote a subset of the sample space. Then R is called "best critical region" of size α for testing the simple hypothesis $H_0 : \theta = \theta'$ against $H_a : \theta = \theta''$ if for every subset D of the sample space for which $P[(X_1, \ldots, X_n) \in D|H_0] = \alpha$ it is true that

   a. $P[(X_1, \ldots, X_n) \in R|H_0] = \alpha$.

   b. $P[(X_1, \ldots, X_n) \in R|H_a] \geq P[(X_1, \ldots, X_n) \in D|H_a]$.

   Explanation:
   In general, there are many subsets D for which $P[(X_1, \ldots, X_n) \in D|H_0] = \alpha$, but there is one of these subsets, denoted with R, such that the power of the test associated with R is larger than any other subset D.

2. Example:
   Suppose $X \sim b(5, p)$. We want to test $H_0 : p = \frac{1}{2}$ against $H_a : p = \frac{3}{4}$ using one random value of X. We list all the probabilities of $b(5, \frac{1}{2})$ and $b(5, \frac{3}{4})$ in the next table:

x	0	1	2	3	4	5
$P(X = x \vert p = \frac{1}{2})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$P(X = x \vert p = \frac{3}{4})$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

Suppose we decided to use $\alpha = \frac{1}{32}$. We want to find the best critical region of size $\alpha = \frac{1}{32}$. We observe that $P(X = 0|p = \frac{1}{2}) = \frac{1}{32}$ and $P(X = 5|p = \frac{1}{2}) = \frac{1}{32}$. Therefore, there are two subsets $D_1(x = 0)$ and $D_2(x = 5)$, for which $P(X \in D_1|H_0) = \frac{1}{32}$ and $P(X \in D_2|H_0) = \frac{1}{32}$. One of these subsets will be our best critical region. Which one of these two subsets has the largest power? We compute: $P(X = 0|p = \frac{3}{4}) = \frac{1}{1024}$ and $P(X = 5|p = \frac{3}{4}) = \frac{243}{1024}$, therefore the best critical region of size $\alpha = \frac{1}{32}$ is $R = \{x = 5\}$.

We also observe that the best critical region of size $\alpha = \frac{1}{32}$ corresponds to the point in D for which $\frac{P(X=x|p=\frac{1}{2})}{P(X=x|p=\frac{3}{4})}$ is the minimum. We see this in the next table where we compute the ratios $\frac{P(X=x|p=\frac{1}{2})}{P(X=x|p=\frac{3}{4})}$.

x	0	1	2	3	4	5
$P(X = x\|p = \frac{1}{2}$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$P(X = x\|p = \frac{3}{4}$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{32}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$
$\frac{P(X=x\|p=\frac{1}{2})}{P(X=x\|p=\frac{3}{4})}$	32	$\frac{32}{3}$	$\frac{32}{9}$	$\frac{32}{27}$	$\frac{32}{81}$	$\frac{32}{243}$

Another example: Suppose $\alpha = \frac{6}{32}$. Find the best critical region of size $\alpha = \frac{6}{32}$.

3. Neyman-Pearson theorem:
   Suppose X is a random variable and we need to decide whether the probability distribution is either $f_0(x)$ or $f_1(x)$. For example, we may want to test that $f_0(x)$ is $N(18, 1)$ against the alternative that $f_1(x)$ is $N(28, 1)$.

   Let k be some positive number, and define the following two sets:

   $$A = \left\{ x \,\middle|\, \frac{f_0(x)}{f_1(x)} > k \right\}$$

   and

   $$R = \left\{ x \,\middle|\, \frac{f_0(x)}{f_1(x)} < k \right\}$$

   The Neyman-Pearson decision rule is the following:
   If data x is in set A, then accept H_0.
   If data x is in set R, then accept H_a.

   Let α be the probability of Type I error based on A and R above.

   Therefore for the Neyman-Pearson Lemma we have:

   $$\alpha = \int_R f_0(x)dx \text{ and } 1 - \alpha = \int_A f_0(x)dx,$$

   and

   $$\beta = \int_A f_1(x)dx$$

   Suppose that there is a competitor test with acceptance region A^* and rejection region R^*, such that $\alpha^* \leq \alpha$.

   Therefore for this competitor test we have:

   $$\alpha^* = \int_{R^*} f_0(x)dx \text{ and } 1 - \alpha^* = \int_{A^*} f_0(x)dx,$$

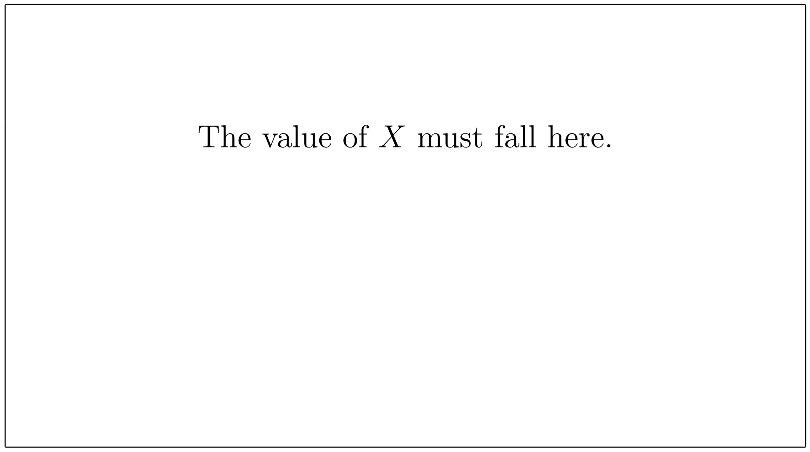
   and

   $$\beta^* = \int_{A^*} f_1(x)dx$$

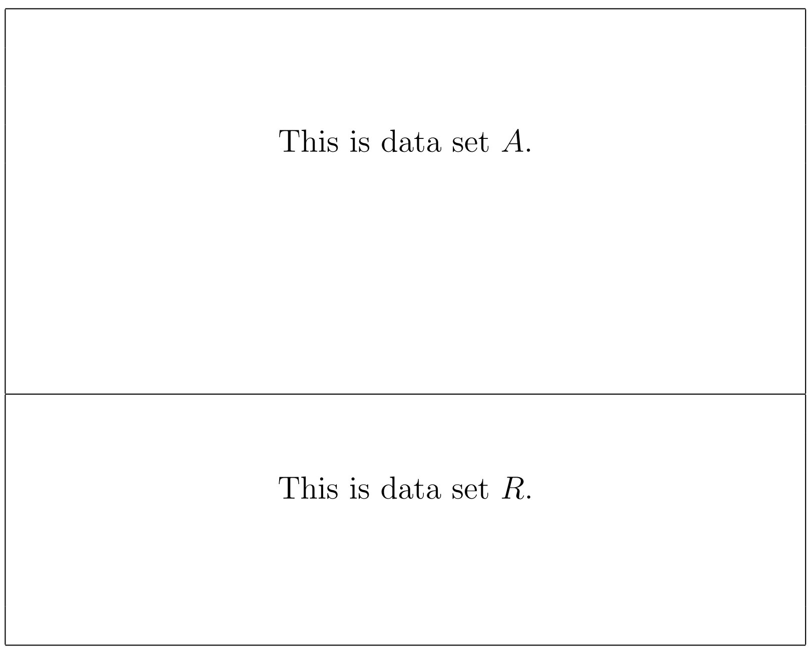
   The Neyman-Pearson Lemma claims that this test is the best, in the sense that any other competitor test with Type I error α^* such that $\alpha^* \leq \alpha$ will have higher probability of Type II error. Therefore, $\beta^* - \beta \geq 0$.

Proof:

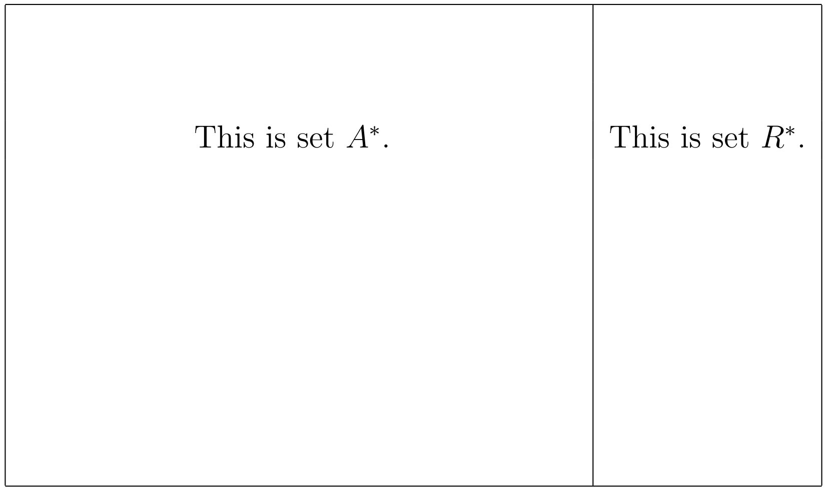
This is the entire data space:



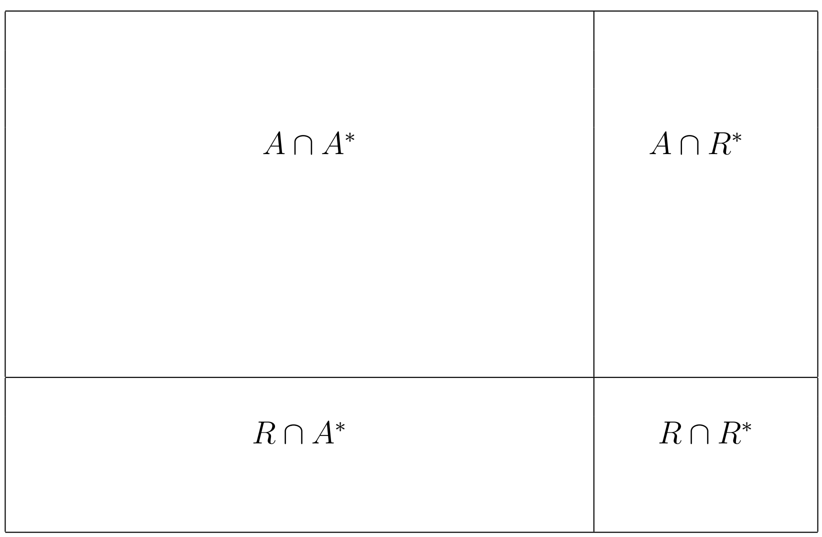
This is the data space partitioned by the Neyman-Pearson Lemma:



This is the data space partitioned by the competitor test:



This is the final picture showing the partitioned of the data space based on the two tests:



We need to calculate $\beta^* - \beta$:

$$\begin{aligned}
\beta^* - \beta &= \int_{A^*} f_1(x)dx - \int_A f_1(x)dx \\
&= \int_{(A^* \cap A) \cup (A^* \cap R)} f_1(x)dx - \int_{(A \cap A^*) \cup (A \cap R^*)} f_1(x)dx \\
&= \int_{A^* \cap A} f_1(x)dx + \int_{A^* \cap R} f_1(x)dx - \left\{ \int_{A \cap A^*} f_1(x)dx + \int_{A \cap R^*} f_1(x)dx \right\} \\
&= \int_{A^* \cap R} f_1(x)dx - \int_{A \cap R^*} f_1(x)dx.
\end{aligned}$$

For the first integral $\int_{A^* \cap R} f_1(x)dx$, it should be true that $\frac{f_0(x)}{f_1(x)} < k$ because it is done over the subset R. Therefore the integral will be smaller if we replace $f_1(x)$ with $\frac{f_0(x)}{k}$ since $f_1(x) > \frac{f_0(x)}{k}$.

For the second integral $\int_{A \cap R^*} f_1(x)dx$, it should be true that $\frac{f_0(x)}{f_1(x)} > k$ because it is done over the subset A. Therefore the integral will be larger if we replace $f_1(x)$ with $\frac{f_0(x)}{k}$ since $f_1(x) < \frac{f_0(x)}{k}$.

These two changes above will give us:

$$\begin{aligned}\beta^* - \beta &\geq \int_{A^* \cap R} \frac{1}{k} f_0(x)dx - \int_{A \cap R^*} \frac{1}{k} f_0(x)dx \\ &= \frac{1}{k} \int_{A^* \cap R} f_0(x)dx - \frac{1}{k} \int_{A \cap R^*} f_0(x)dx.\end{aligned}$$

Now, add an subtract $\frac{1}{k} \int_{A \cap A^*} f_0(x)dx$ to get:

$$\begin{aligned}\beta^* - \beta &\geq \frac{1}{k} \int_{A^* \cap R} f_0(x)dx + \frac{1}{k} \int_{A \cap A^*} f_0(x)dx \\ &- \frac{1}{k} \int_{A \cap R^*} f_0(x)dx - \frac{1}{k} \int_{A \cap A^*} f_0(x)dx.\end{aligned}$$

Because, $A^* = (A^* \cap A) \cup (A^* \cap R)$ and $A = (A \cap A^*) \cup (A \cap R^*)$ we finally get:

$$\begin{aligned}\beta^* - \beta &\geq \frac{1}{k} \int_{A^*} f_0(x)dx - \frac{1}{k} \int_{A} f_0(x)dx \\ &\geq \frac{1}{k}(1-\alpha^*) - \frac{1}{k}(1-\alpha) \\ &\geq \frac{1}{k}(\alpha - \alpha^*) \geq 0.\end{aligned}$$

Therefore, the competitor test with equal or better Type I error probability must have larger Type II error probability.

4. Neyman-Pearson theorem. Summary and examples:
Suppose we wish to test the simple hypothesis
$H_0 : \theta = \theta_0$
against the alternative simple hypothesis
$H_a : \theta = \theta_a$.
As always, a sample of $X_1, X_2, \ldots, X_n$ is selected from a probability distribution with unknown parameter θ. Let $L(\theta_0)$ denote the likelihood function when $\theta = \theta_0$ and $L(\theta_a)$ denote the likelihood function when $\theta = \theta_a$. Then for a given significance level α, the test that maximizes the power has a rejection region determined by $\frac{L(\theta_0)}{L(\theta_a)} < k$, where k is some constant. This test will be the most powerful test for testing H_0 against H_a.

The previous result applies to simple hypotheses. Usually one of the two hypotheses is composite. For example: $H_0 : \theta = \theta_0$
against the alternative composite hypothesis
$H_a : \theta > \theta_0$.
We say that a test that is most powerful for every simple alternative in H_a is uniformly most powerful.

Example 1:
Let X be a single observation from the probability density function $f(x) = \theta x^{\theta-1}, 0 < x < 1$. Find the most powerful test using significance level $\alpha = 0.05$ for testing
$H_0 : \theta = 1$
$H_a : \theta = 2$.

Example 2:
Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma)$, with known σ^2. Find the uniformly most powerful test using significance level α for testing
$H_0 : \mu = \mu_0$
$H_a : \mu > \mu_0$.

Example 3:
Let $X \sim exp(\frac{1}{\lambda})$. Therefore, $f(x) = \frac{1}{\lambda}e^{-\frac{1}{\lambda}x}, \lambda > 0, x > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution.

a. Show that the best critical region for testing
   $H_0 : \lambda = 3$
   $H_a : \lambda = 5$
   is based on $\sum_{i=1}^{n} x_i$.

b. If $n = 12$ and using $\frac{2}{\lambda}\sum_{i=1}^{n} x_i \sim \chi^2_{24}$ find the best critical region when the significance level $\alpha = 0.05$.

Example 4:
Suppose X has the possible values 0,1,2,3,4. Suppose that the null hypothesis says that X is uniform on these integers, while the alternative hypothesis says that $X \sim b(4, \frac{1}{2})$. Let's see what happens if we let k of the Neyman-Pearson lemma be equal to 0.6. Complete the next table and find the best critical region when $k = 0.6$ and compute the power of the test.

x	0	1	2	3	4
$P(X = x \mid H_0)$					
$P(X = x \mid H_a)$					
$\frac{P(X=x \mid H_0)}{P(X=x \mid H_a)}$					