Confidence intervals: large sample theory for maximum likelihood estimates

Another method of constructing confidence intervals is based on the large sample theory of maximum likelihood estimates.

As the sample size $n$ increases it can be shown that the maximum likelihood estimate $\hat{\theta}$ of a parameter $\theta$ follows approximately normal distribution with mean $\theta$ and variance equal to the lower bound of the Cramer-Rao inequality.

$$\hat{\theta} \sim N \left( \theta, \sqrt{\frac{1}{nI(\theta)}} \right), \quad \text{where} \quad \sqrt{\frac{1}{nI(\theta)}} \quad \text{is the lower bound of the Cramer-Rao inequality.}$$

Because $I(\theta)$ (Fisher’s information - see previous handout) is a function of the unknown parameter $\theta$ we replace $\theta$ with its maximum likelihood estimate $\hat{\theta}$ to get $I(\hat{\theta})$.

Since,

$$Z = \frac{\hat{\theta} - \theta}{\sqrt{\frac{1}{nI(\theta)}}},$$

we can write

$$P\left(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}\right)$$

We replace $Z$ with $Z = \frac{\hat{\theta} - \theta}{\sqrt{\frac{1}{nI(\theta)}}}$ to get

$$P\left(-z_{\frac{\alpha}{2}} \leq \frac{\hat{\theta} - \theta}{\sqrt{\frac{1}{nI(\theta)}}} \leq z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

And finally,

$$P\left(\hat{\theta} - z_{\frac{\alpha}{2}}\sqrt{\frac{1}{nI(\theta)}} \leq \theta \leq \hat{\theta} + z_{\frac{\alpha}{2}}\sqrt{\frac{1}{nI(\theta)}}\right)$$

Therefore we are $1 - \alpha$ confident that $\theta$ falls in the interval

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{1}{nI(\theta)}}$$
Example:
Use the result above to construct a confidence interval for the Poisson parameter $\lambda$. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables from a Poisson distribution with parameter $\lambda$.

We know that the maximum likelihood estimate of $\lambda$ is $\hat{\lambda} = \bar{x}$. We need to find the lower bound of the Cramer-Rao inequality:

$$f(x) = \frac{\lambda^xe^{-\lambda}}{x!} \Rightarrow ln f(x) = xln\lambda - \lambda - ln x!$$

Let’s find the first and second derivatives w.r.t. $\lambda$.

$$\frac{\partial ln f(x)}{\partial \lambda} = \frac{x}{\lambda} - 1 \quad \text{and} \quad \frac{\partial^2 ln f(x)}{\partial \lambda^2} = -\frac{x}{\lambda^2}.$$ 

Therefore,

$$\frac{1}{-nE\left(\frac{\partial^2 ln f(x)}{\partial \lambda^2}\right)} = \frac{1}{-nE\left(-\frac{X}{X^2}\right)} = \frac{\lambda^2}{n\lambda} = \frac{\lambda}{n}.$$ 

Therefore when $n$ is large $\hat{\lambda}$ follows approximately

$$\hat{\lambda} \sim N\left(\lambda, \sqrt{\frac{\lambda}{n}}\right)$$

Because $\lambda$ is unknown we replace it with its mle estimate $\hat{\lambda}$:

$$\hat{\lambda} \sim N\left(\lambda, \sqrt{\frac{\lambda}{n}}\right) \quad \text{or} \quad \hat{\lambda} \sim N\left(\lambda, \sqrt{\frac{\bar{X}}{n}}\right)$$

Therefore, the confidence interval for $\lambda$ is:

$$\bar{X} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}$$

Application:
The number of pine trees at a certain forest follows the Poisson distribution with unknown parameter $\lambda$ per acre. A random sample of size $n = 50$ acres is selected and the number of pine trees in each acre is counted. Here are the results:

$$7 4 5 3 1 5 7 6 4 3 2 6 6 9 2 3 3 7 2 5 5 4 4 8 8 7 2 6 3 5 0$$
$$5 8 9 3 4 5 4 6 1 0 5 4 6 3 6 9 5 7 6$$

The sample mean is $\bar{x} = 4.76$. Therefore a 95% confidence interval for the parameter $\lambda$ is

$$4.76 \pm 1.96\sqrt{\frac{4.76}{50}} \quad \text{or} \quad 4.76 \pm 0.31$$

Therefore $4.15 \leq \lambda \leq 5.34$. 