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Statistics 100B **Instructor: Nicolas Christou**

Moment generating functions

Definition:

$$M_X(t) = Ee^{tX}$$

Therefore,
If X is discrete

$$M_X(t) = \sum_x e^{tX} p(x)$$

If X is continuous

$$M_X(t) = \int_x e^{tX} f(x)dx$$

Aside:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Similarly,

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots$$

Let X be a discrete random variable.

$$M_X(t) = \sum_x e^{tX} p(x) = \sum_x \left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right] p(x)$$

or

$$M_X(t) = \sum_x p(x) + \frac{t}{1!}\sum_x xp(x) + \frac{t^2}{2!}\sum_x x^2 p(x) + \frac{t^3}{3!}\sum_x x^3 P(x) + \cdots$$

To find the k_{th} moment simply evaluate the k_{th} derivative of the $M_X(t)$ at $t = 0$.

$$EX^k = [M_X(t)]_{t=0}^{k_{th}\text{derivative}}$$

For example:
First moment:

$$M_X(t)' = \sum_x xp(x) + \frac{2t}{2!}\sum_x x^2 p(x) + \cdots$$

We see that $M_X(0)' = \sum_x xp(x) = E(X)$.

Similarly,
Second moment

$$M_X(t)'' = \sum_x x^2 p(x) + \frac{6t}{3!} \sum_x x^3 p(x) + \cdots$$

We see that $M_X(0)'' = \sum_x x^2 p(x) = E(X^2)$.

Examples:

1. Find the moment generating function of $X \sim b(n, p)$.

$$\begin{aligned}
M_X(t) &= Ee^{tx} \\
M_X(t) &= \sum_x e^{tx} p(x) \\
M_X(t) &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
M_X(t) &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\
\text{Using the binomial theorem } (a+b)^n &= \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} \text{ we get} \\
M_X(t) &= (pe^t + 1 - p)^n
\end{aligned}$$

2. Find the moment generating function of $X \sim Poisson(\lambda)$.

$$\begin{aligned}
M_X(t) &= Ee^{tx} \\
M_X(t) &= \sum_x e^{tx} p(x) \\
M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \\
M_X(t) &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
M_X(t) &= e^{-\lambda} e^{\lambda e^t} \\
M_X(t) &= e^{\lambda(e^t - 1)}
\end{aligned}$$

3. Find the moment generating function of $X \sim \Gamma(\alpha, \beta)$. We say that X follows a gamma distribution with parameters α, β if its pdf is given by $f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}$, $x > 0, \alpha > 0, \beta > 0$, where $\Gamma(\alpha)$ is the gamma function defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

$$\begin{aligned}
M_X(t) &= Ee^{tx} \\
M_X(t) &= \int_x e^{tx} f(x)
\end{aligned}$$

$$\begin{aligned}
M_X(t) &= \int_{x=0}^{\infty} e^{tx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx \\
M_X(t) &= \int_{x=0}^{\infty} \frac{x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)}}{\Gamma(\alpha)\beta^\alpha} dx \quad \text{Use the transformation } y = x(\frac{1}{\beta} - t) \text{ to get} \\
&\vdots \\
M_X(t) &= (1-\beta t)^{-\alpha}
\end{aligned}$$

4. Find the moment generating function of $X \sim exp(\lambda)$. The exponential distribution is a special case of $\Gamma(\alpha, \beta)$ with $\alpha = 1$ and $\beta = \frac{1}{\lambda}$ (why?), therefore, $M_X(t) = (1 - \frac{t}{\lambda})^{-1}$.

5. Find the moment generating function of $Z \sim N(0, 1)$. Aside note: The normal distribution $X \sim N(\mu, \sigma)$ has pdf

   $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2},$$

   and the standard normal $Z \sim N(0, 1)$ has pdf

   $$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

   Therefore,

   $$\begin{aligned}
   M_Z(t) &= Ee^{tz} \\
   M_Z(t) &= \int_z e^{tz} f(z) \\
   M_Z(t) &= \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
   \text{Add/subtract to the exponent } \frac{1}{2}t^2 & \\
   M_Z(t) &= e^{\frac{1}{2}t^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz \\
   \text{Explain why } \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz &= 1 \\
   \text{Therefore,} & \\
   M_Z(t) &= e^{\frac{1}{2}t^2}
   \end{aligned}$$

Theorem:
Let X, Y be independent random variables with moment generating functions $M_X(t), M_Y(t)$ respectively. Then, the moment generating function of the sum of these two random variables is equal to the product of the individual moment generating functions:

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

Proof:

Examples: Let X, Y independent random variables. Use this theorem to find the distribution of $X + Y$:

a. $X \sim b(n_1, p), Y \sim b(n_2, p)$.

b. $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2)$.

Properties of moment generating functions:
Let X be a random variable with moment generating function $M_X(t) = Ee^{tX}$, and a, b are constants

1. $M_{X+a}(t) = e^{at} M_X(t)$

2. $M_{bX}(t) = M_X(bt)$

3. $M_{\frac{X+a}{b}} = e^{\frac{a}{b}t} M_X(\frac{t}{b})$

Use these properties and the moment generating function of $Z \sim N(0, 1)$ to find the moment generating function of $X \sim N(\mu, \sigma)$

Find the distribution of $X + Y$, where $X \sim N(\mu_1, \sigma_1), Y \sim N(\mu_2, \sigma_2)$.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from $N(\mu, \sigma)$. Use moment generating functions to find the distribution of

a. $T = X_1 + X_2 + \ldots + X_n$.

b. $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.

Distribution of the sample mean - Sampling from normal distribution

If we sample from normal distribution $N(\mu, \sigma)$ then $\bar{X}$ follows exactly the normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ regardless of the sample size n. In the next figure we see the effect of the sample size on the shape of the distribution of $\bar{X}$. The first figure is the $N(5, 2)$ distribution. The second figure represents the distribution of $\bar{X}$ when $n = 4$. The third figure represents the distribution of $\bar{X}$ when $n = 16$.