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Statistics 100B **Instructor: Nicolas Christou**

Noncentral χ^2 and noncentral F distributions

Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables with $Y_i \sim N(\mu_i, \sigma^2), i = 1, 2, \ldots, n$. If each $\mu_i = 0$ then $Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2} \sim \chi^2_n$. What if each $\mu_i \neq 0$?

The m.g.f. of Q is given by

$$M_Q(t) = E\left[exp\left(t\sum_{i=1}^n \frac{Y_i^2}{\sigma^2}\right)\right] = E\left[exp\left(t\frac{Y_1^2}{\sigma^2}\right)\right] \times E\left[exp\left(t\frac{Y_2^2}{\sigma^2}\right)\right] \times \ldots \times E\left[exp\left(t\frac{Y_n^2}{\sigma^2}\right)\right]$$

Let's examine one of these expectations:

$$E\left[exp\left(t\frac{Y_i^2}{\sigma^2}\right)\right] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{ty_i^2}{\sigma^2} - \frac{(y_i-\mu_i)^2}{2\sigma^2}\right] dy_i.$$

Evaluate the integral using:

$$\begin{aligned}
\frac{ty_i^2}{\sigma^2} - \frac{(y_i-\mu_i)^2}{2\sigma^2} &= -\frac{y_i^2(1-2t)}{2\sigma^2} + \frac{2\mu_i y_i}{2\sigma^2} - \frac{\mu_i^2}{2\sigma^2} \\
&= \frac{t\mu_i^2}{\sigma^2(1-2t)} - \frac{1-2t}{2\sigma^2}\left(y_i - \frac{\mu_i}{1-2t}\right)^2.
\end{aligned}$$

Back to the expectation

$$E\left[exp\left(t\frac{Y_i^2}{\sigma^2}\right)\right] = exp\left[\frac{t\mu_i^2}{\sigma^2(1-2t)}\right] \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \times exp\left[-\frac{1-2t}{2\sigma^2}\left(y_i - \frac{\mu_i}{1-2t}\right)^2\right] dy_i.$$

If we multiply and divide by $\sqrt{1-2t}$ we have the integral of a normal pd.f. with mean $\frac{\mu_i}{1-2t}$ and variance $\frac{\sigma^2}{1-2t}$ (and therefore it is equal to 1, to finally get

$$E\left[exp\left(t\frac{Y_i^2}{\sigma^2}\right)\right] = \frac{1}{\sqrt{1-2t}} exp\left[\frac{t\mu_i^2}{\sigma^2(1-2t)}\right].$$

Now we can find the moment generating function of $Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2}$.

$$M_Q(t) = (1-2t)^{-\frac{n}{2}} exp\left[\frac{t\sum_{i=1}^n \mu_i^2}{\sigma^2(1-2t)}\right].$$

In general, a random variable Q that has m.g.f. of the form

$$M_Q(t) = (1-2t)^{-\frac{n}{2}} e^{\theta\frac{t}{1-2t}}$$

follows the χ^2 distribution with noncentrality parameter θ. We write $Q \sim \chi^2(n, \theta)$. Therefore

$$Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2} \sim \chi^2\left(n, \sum_{i=1}^n \frac{\mu_i^2}{\sigma^2}\right).$$

Note: If the noncentrality parameter is zero then $Q \sim \chi^2_n$.