University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Random vectors and properties

Mean and variance of a random vector

Let $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ be a random vector with $E\mathbf{Y} = \begin{pmatrix} EY_1 \\ EY_2 \\ \vdots \\ EY_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \boldsymbol{\mu}$. The variance covariance matrix of $\mathbf{Y}$ denoted with $\boldsymbol{\Sigma} = \mathrm{var}(\mathbf{Y})$ is defined as follows:

$$\begin{aligned}
\mathrm{var}(\mathbf{Y}) &= E(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})' \\
&= E\begin{pmatrix} Y_1-\mu_1 \\ Y_2-\mu_2 \\ \vdots \\ Y_n-\mu_n \end{pmatrix}(Y_1-\mu_1, Y_2-\mu_2, \ldots, Y_n-\mu_n) \\
&= E\begin{pmatrix} (Y_1-\mu_1)^2 & (Y_1-\mu_1)(Y_2-\mu_2) & \ldots & (Y_1-\mu_1)(Y_n-\mu_n) \\ (Y_2-\mu_2)(Y_1-\mu_1) & (Y_2-\mu_2)^2 & \ldots & (Y_2-\mu_2)(Y_n-\mu_n) \\ \vdots & \vdots & \ddots & \vdots \\ (Y_n-\mu_n)(Y_1-\mu_1) & (Y_n-\mu_n)(Y_2-\mu_2) & \ldots & (Y_n-\mu_n)^2 \end{pmatrix}
\end{aligned}$$

Take now expectation for each element of the matrix above. What do we get?

So $\boldsymbol{\Sigma}$ is the variance covariance matrix of the vector $\mathbf{Y}$. It is symmetric and positive definite.

Suppose $Y_1, \ldots, Y_n$ are independent identically distributed (i.i.d.) random variables. This means that $E[Y_i] = \mu, i = 1, \ldots, n$, $\mathrm{var}[Y_i] = \sigma^2, i = 1, \ldots, n$ and $\mathrm{cov}[Y_i, Y_j] = 0, i \neq j$. Find the expression of the mean vector of $\mathbf{Y}$ and the variance covariance matrix of $\mathbf{Y}$ using the vector $\mathbf{1} = (1, 1, \ldots, 1)'$ and the identity matrix $\mathbf{I}$ for this special case.

Two important results are given below: The mean and variance of a linear combination of the elements of a random vector and the mean and variance of a set of linear combinations of the elements of a random vector. In the first case we will examine $\mathbf{a}'\mathbf{Y} = a_1Y_1 + \ldots + a_nY_n$ while in the second case we will examine a set of p of these combinations.

1. Expected value and variance of a linear combination of $\mathbf{Y}$. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ be a vector of constants and let $q = \mathbf{a}'\mathbf{Y}$. Then $E(q) = E(\mathbf{a}'\mathbf{Y}) = \mathbf{a}'E(\mathbf{Y}) = \mathbf{a}'\boldsymbol{\mu}$. The variance of q can be found as follows:

$$\begin{aligned} var(q) = E(q-\mu_q)^2 &= E(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu})^2 \\ &= E(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu})(\mathbf{a}'\mathbf{Y} - \mathbf{a}'\boldsymbol{\mu}) \\ &= \mathbf{a}'E(\mathbf{Y} - \boldsymbol{\mu})(\mathbf{Y} - \boldsymbol{\mu})'\mathbf{a} \\ &= \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}. \end{aligned}$$

Note: q is a scalar and therefore its variance should be a scalar and not a matrix. We can verify that $var(q) = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$ is 1×1.

We can also express the variance of a linear combination using summation notation as follows:

$$\begin{aligned} \mathrm{var}\left(\sum_{i=1}^{n} a_iY_i\right) &= \mathrm{cov}\left(\sum_{i=1}^{n} a_iY_i, \sum_{j=1}^{n} a_jY_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\mathrm{cov}(Y_i, Y_j) \\ &= \sum_{i=1}^{n} a_i^2\mathrm{var}(Y_i) + \sum_{i=1}^{n}\sum_{j\neq i}^{n} a_ia_j\mathrm{cov}(Y_i, Y_j) \\ &= \sum_{i=1}^{n} a_i^2\mathrm{var}(Y_i) + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} a_ia_j\mathrm{cov}(Y_i, Y_j) \end{aligned}$$

Example;

Let $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 4 \end{pmatrix}$, and $\boldsymbol{\Sigma} = \begin{pmatrix} 3 & 2 & 3 & 3 \\ 2 & 5 & 5 & 4 \\ 3 & 5 & 9 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Find the mean and variance of $q = 4Y_1 - 2Y_2 + Y_3 + 3Y_4$.

2. Let $\mathbf{A}$ be a $p \times n$ matrix of constants. We will examine now $\mathbf{Q} = \mathbf{AY}$. Unlike q (see result (1) above), $\mathbf{Q}$ is a $p \times 1$ vector and therefore its variance should be a $p \times p$ matrix. Let's find the expected value of $\mathbf{Q}$ first. $E(\mathbf{Q}) = E(\mathbf{AY}) = \mathbf{A}E(\mathbf{Y}) = \mathbf{A}\boldsymbol{\mu}$. For the variance of $\mathbf{Q}$ use the definition of the variance covariance matrix of a random vector.

$$var(\mathbf{Q}) =$$

3. **Expectation of a quadratic expression**
   Let $\mathbf{Y}$ be a random vector $n \times 1$ and let $\mathbf{A}$ be an $n \times n$ matrix of constants. Consider the quadratic expression $\mathbf{Y}'\mathbf{AY}$. We want to find the expected value of this quadratic expression. Such expressions appear in linear models and the result here provides a method for finding such expectations. For example, suppose $n = 2$. Find the expected value of $\begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 2Y_1^2 + 3Y_2^2 + 9Y_1Y_2$

   To find the expected value of this quadratic expression we will use properties of the trace of a square matrix. We can do this because $\mathbf{Y}'\mathbf{AY}$ is a scalar. We will also need this result: $E[\mathbf{YY}'] = \boldsymbol{\Sigma} + \boldsymbol{\mu\mu}'$. (Show this result using the definition of the variance covariance matrix).

$$E[\mathbf{Y}'\mathbf{AY}] = E[tr(\mathbf{Y}'\mathbf{AY})]$$

4. Other results:

   a. Covariance between two linear combinations:
   $\text{cov}(\mathbf{a}'\mathbf{Y}, \mathbf{b}'\mathbf{Y}) = \mathbf{a}\boldsymbol{\Sigma}\mathbf{b}'$. This is a scalar.
   Example;
   Let $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 4 \end{pmatrix}$, and $\boldsymbol{\Sigma} = \begin{pmatrix} 3 & 2 & 3 & 3 \\ 2 & 5 & 5 & 4 \\ 3 & 5 & 9 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Find the covariance between $q_1 = 4Y_1 - 2Y_2 + Y_3 + 3Y_4$ and $q_2 = Y_1 + 3Y_2 - 5Y_3 - 4Y_4$.

   b. $\text{cov}(\mathbf{AY}, \mathbf{BY}) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}'$. This is a matrix.